Eur. Phys. J. A **29**, 289–306 (2006)
DOI 10.1140/epja/i2005-10315-8

THE EUROPEAN PHYSICAL JOURNAL A

Spin-$\frac{3}{2}$ beyond the Rarita-Schwinger framework

M. Napsuciale[1,a], M. Kirchbach[2], and S. Rodriguez[2]

[1] Insituto de Fisica, Universidad de Guanajuato, Lomas del Bosque 103, Fracc. Lomas del Campestre, 37150 Leon, Guanajuato, Mexico
[2] Instituto de Fisica, UASLP, Av. Manuel Nava 6, Zona Universitaria, San Luis Potosi, SLP 78290, Mexico

Received: 15 October 2005 / Revised: 29 August 2006 /
Published online: 20 September 2006 –
Communicated by U.-G. Meißner

Abstract. We employ the two independent Casimir operators of the Poincaré group, the squared four-momentum, p^2, and the squared Pauli-Lubanski vector, $\mathcal{W}^2$, in the construction of a covariant mass m, and spin-$\frac{3}{2}$ projector in the four-vector spinor, ψ_μ. This projector provides the basis for the construction of an interacting Lagrangian that describes a causally propagating spin-$\frac{3}{2}$ particle coupled to the electromagnetic field by a gyromagnetic ratio of $g_{\frac{3}{2}} = 2$.

PACS. 11.30.Cp Lorentz and Poincaré invariance – 03.65.Pm Relativistic wave equations

1 Introduction

High-spin particles occupy an important place in theoretical physics. For the first time they were observed as resonant excitations in pion-nucleon scattering. The Particle Data Group [1] lists more than thirty non-strange baryon resonances with spins ranging from $\frac{3}{2}$ to $\frac{15}{2}$, and more than twenty strange ones with spins from $\frac{3}{2}$ to $\frac{9}{2}$. Baryon resonances have been extensively investigated in the past among others at the former Los Alamos Meson Physics Facility (LAMPF), and at present their study continues at the Thomas Jefferson National Accelerator Facility (TJNAF) [2]. Such particles are of high relevance in the description of photo- and electro-pion production off proton, where they appear as intermediate states, studies to which the Mainz Microtron (MAMI) devotes itself since many years [3]. Search for high-spin solutions to the QCD Lagrangian has been recently reported by the Lattice Collaboration in ref. [4]. Moreover, also the twistor formalism has been employed in the construction of high-spin fields [5]. Integer high-spin meson resonances with spins ranging from 0 to 6 can have importance in various processes revealing the fundamental features of QED at high energies such as pair production [6]. However, the most attractive high-spin fields appear in proposals for physics beyond the standard model which invoke supersymmetry [7] and contain gauge fields of fractional spins such as the gravitino —the supersymmetric partner of the ordinary spin-2 graviton. Supersymmetric theories open the venue to the production of fundamental spin-$\frac{3}{2}$ particles at early stages of the universe, whose understanding can play an important role in its evolution [8].

The description of high spins takes its origin from refs. [9–11] which suggest to consider any fractional spin s as the highest spin in the traceless and totally symmetric rank-$(s-\frac{1}{2})$ Lorentz tensor with Dirac spinor components, $\psi_{\mu_1\ldots\mu_{s-\frac{1}{2}}}$. For spin-$\frac{3}{2}$ one has to consider the four-vector spinor, ψ_μ,

$$\psi_\mu = A_\mu \otimes \psi \simeq \left(\frac{1}{2},\frac{1}{2}\right) \otimes \left[\left(\frac{1}{2},0\right) \oplus \left(\frac{1}{2},0\right)\right], \quad (1)$$

the direct product between the four-vector, A_μ, and the Dirac spinor, ψ, and solve the system of three linear (in the momenta) equations

$$(\not p - m)\psi_\mu = 0, \quad (2)$$
$$\gamma^\mu \psi_\mu = 0, \quad (3)$$
$$p^\mu \psi_\mu = 0, \quad (4)$$

known as the Rarita-Schwinger (RS) framework. Next, one designs [12] the most general family of Lagrangians depending on the undetermined parameter (A), with the aim to reproduce eqs. (2)-(4). The Lagrangians obtained this way read

$$\mathcal{L}^{(\mathrm{RS})}(A) = \overline{\psi}^\mu \left(p_\alpha \Gamma_\mu{}^{\alpha}{}_\nu(A) - m\, B_{\mu\nu}(A)\right)\psi^\nu, \quad (5)$$

[a] e-mail: mauro@fisica.ugto.mx

where

$$\begin{aligned}\Gamma_\mu{}^\alpha{}_\nu(A) &= g_{\mu\nu}\gamma_\alpha + A(\gamma_\mu g^\alpha_\nu + g^\alpha_\mu\gamma_\nu) + B\gamma_\mu\gamma^\alpha\gamma_\nu,\\ B_{\mu\nu}(A) &= g_{\mu\nu} - C\gamma_\mu\gamma_\nu,\\ A \neq \frac{1}{2},\quad B &\equiv \frac{3}{2}A^2 + A + \frac{1}{2},\quad C \equiv 3A^2 + 3A + 1.\end{aligned} \tag{6}$$

The case $A = -\frac{1}{3}$ corresponds to the Lagrangian originally proposed in [10]. Another value widely used in the literature is $A = -1$ in which case the Lagrangian simplifies to

$$\mathcal{L}^{(\mathrm{RS})}(A=-1) = \overline{\psi}^\mu \left(p_\alpha\epsilon_\mu{}^\alpha{}_{\nu\rho}\gamma^5\gamma^\rho - im\,\sigma_{\mu\nu}\right)\psi^\nu. \tag{7}$$

If we define

$$K_{\mu\nu}(A) = p_\alpha\Gamma_\mu{}^\alpha{}_\nu(A) - m\,B_{\mu\nu}(A), \tag{8}$$

the above Lagrangian factorizes as

$$\mathcal{L}^{(\mathrm{RS})}(A) = \overline{\psi}^\mu\, R_{\mu\rho}\left(\frac{A}{2}\right)\, K^{\rho\sigma}(0)R_{\sigma\nu}\left(\frac{A}{2}\right)\psi^\nu, \tag{9}$$

where

$$R_{\mu\rho}(w) \equiv g_{\mu\rho} + w\gamma_\mu\gamma_\rho. \tag{10}$$

This factorization can be used to show that the Lagrangian is invariant under the point transformations

$$\psi_\mu \to \psi'_\mu = R_{\mu\nu}(w)\psi^\nu, \qquad A \to \frac{A-2w}{1+4w}. \tag{11}$$

Over the years, eqs. (2)-(4) have been widely applied in hadron physics to the description of predominantly the $\Delta(1232)$ —and occasionally the $D_{13}(1520)$ resonances and their contributions to various processes. Recent applications of the Rarita-Schwinger spin-$\frac{3}{2}$ description to calculations of light-hadron properties along the line of chiral perturbation theory can be found in refs. [13,14].

The freedom represented by the parameter A reflects invariance under "rotations" mixing the two spin-$\frac{1}{2}^+$ and $\frac{1}{2}^-$ sectors residing in the RS representation space besides spin-$\frac{3}{2}$ [15,16]. It can be shown [17] that the elements of the S-matrix do not depend on the parameter A. Yet, this symmetry, when implemented into the interacting theory, introduces ambiguities represented by free parameters, the so-called "off-shell" parameters [15,16,18,19]. This is not to remain the only disadvantage of the RS framework. A detailed study of eqs. (2), (3), and (4) revealed that the Rarita-Schwinger framework suffers some more fundamental weaknesses. The quantization of the interacting spin-$\frac{3}{2}$ field turned out to be inconsistent with Lorentz covariance, an observation reported by Johnson and Sudarshan in ref. [20]. Furthermore, the wavefronts of the *classical* solutions of the Rarita-Schwinger spin-$\frac{3}{2}$ equations were shown to suffer acausal propagation within the electromagnetic environment, an observation due to Velo and Zwanziger [21,22]. This is an old problem and several remedies have been suggested over the years [23–25]. In [24], it was shown that the standard Rarita-Schwinger description allows to avoid the Velo-Zwanziger problem only to the cost of propagating simultaneously twelve degrees of freedom associated with spin-$\frac{1}{2}$, and spin-$\frac{3}{2}$, while in [26] the same was proved in letting the masses of the spin-$\frac{1}{2}$ sectors go to infinity. The more recent ref. [25] suggests two new wave equations in ψ_μ, one of which is linear and local, and the other, quadratic and non-local. The local equation propagates causally all sixteen degrees of freedom in ψ_μ associated with the spin-cascade $(\frac{1}{2}^+, \frac{1}{2}^-, \frac{3}{2}^-)$, but weighted with three different masses. The non-local equation propagates causally the twelve degrees of freedom corresponding to spin-$\frac{1}{2}^-$ and $\frac{3}{2}^-$ treated as mass-degenerate. The latter results indicate that the description of a causal single spin-$\frac{3}{2}$ propagation is beyond the reach of the Rarita-Schwinger framework, an observation reported also in [27].

It is the goal of the present work to construct a single-spin-$\frac{3}{2}$ Lagrangian and associated wave equation such that the wave fronts of its solutions propagate causally within an electromagnetic environment and the spin-$\frac{3}{2}$ particle is coupled to the electromagnetic field through a gyromagnetic ratio of $g_{\frac{3}{2}} = 2$ as required by unitarity in the ultrarelativistic limit [28,29]. The Lagrangian in question is entirely based upon the Poincaré group generators in ψ_μ and the magnetic coupling is identified in a fully covariant fashion. Compared to this, within the Rarita-Schwinger framework the gyromagnetic factor is extracted at the non-relativistic level [12,30,31] or from calculating pion-nucleon bremsstrahlung and a subsequent comparison to low energy theorems [32].

The paper is organized as follows. In the next section we outline the general procedure of pinning down an invariant subspace of mass m and spin s on the example of a generic Lorentz group representation containing two Poincaré invariant spin-sectors, for simplicity. There, we further present the associated second-order (in the momenta) equation of free motion. As a consistency check for our suggested formalism, we re-derive there the Proca equation in applying the procedure to $(\frac{1}{2}, \frac{1}{2})$. In sect. 3 we apply the above procedure to the four-vector spinor and derive the corresponding equation of motion, the associated Lagrangian, and the respective propagator. Section 4 is devoted to the symmetries of the suggested Lagrangian in the massless limit and its relation to "rotations" within the spin-$\frac{1}{2}$ sector. In sect. 5 we introduce electromagnetic interactions. The paper closes with a brief summary and has two appendices.

2 Particle dynamics and Poincaré group invariants

2.1 The Casimir operators $\mathbf{p}^2$ and $\mathcal{W}^2$ and their invariant vector spaces

In the present work we aim to identify spin-$\frac{3}{2}$ directly and in a covariant fashion according to the conventional understanding of a particle as an invariant vector space of the two Casimir invariants of the Poincaré group, the first being the squared four-momentum, p^2, and the second, the squared Pauli-Lubanski vector, $\mathcal{W}^2$. Accordingly, the corresponding states must be labeled by the eigenvalues of these operators (see ref. [33] for details),

$$p^2\Psi^{(m,s)} = m^2\Psi^{(m,s)}, \tag{12}$$
$$\mathcal{W}^2\Psi^{(m,s)} = -p^2 s(s+1)\Psi^{(m,s)}. \tag{13}$$

Here m stands for the mass, while $\Psi^{(m,s)}$ denotes a generic Poincaré group representation of mass m and rest-frame spin s. Equation (12) is the Klein-Gordon equation that fixes the mass of the states, while eq. (13) fixes the spin. As already mentioned in the introduction, the mass shell condition as reflected by the Klein-Gordon equation has been of wide use in the formulation of free-particle Lagrangians, not so the spin condition. We shall formulate a new Lagrangian formalism that incorporates eqs. (12), (13) on equal footing and obtain a single condition on the field that encodes both the mass shell, and spin conditions. In so doing, we first have to resolve the notorious problem of non-coincidence between Poincaré and Lorentz group labels that occurs for all representations beyond $(s,0)\oplus(0,s)$.

Indeed, Lorentz representations are labeled by the so-called left (s_L), and right-handed, (s_R), "spins", the respective eigenvalues to $\boldsymbol{S}_L^2 = \frac{1}{4}(\boldsymbol{J}+i\boldsymbol{K})^2$, and $\boldsymbol{S}_R^2 = \frac{1}{4}(\boldsymbol{J}-i\boldsymbol{K})^2$, where $\boldsymbol{J}$, and $\boldsymbol{K}$ represent the generators of rotations and boost in the basis of interest. The Poincaré s-label enters the Lorentz representation, $\Psi^{(m,(s_L,s_R))}$, via $s = |s_L - s_R|, |s_L - s_R| + 1, \ldots, (s_L + s_R)$, which causes reducibility of $\Psi^{(m,(s_L,s_R))}$ into the following Poincaré invariant subspaces:

$$\Psi^{(m,(s_L,s_R))} \longrightarrow \Psi^{(m,|s_L-s_R|)} \oplus \Psi^{(m,|s_L-s_R|+1)} \oplus \ \ldots \oplus \Psi^{(m,(s_L+s_R))}. \tag{14}$$

The problem one is facing now is the covariant tracking of the sector of interest. In the next subsection we formulate our procedure for the covariant tracking of the highest spin-$\frac{3}{2}$ of mass m in the vector spinor representation ψ_μ.

2.2 Covariant mass-m and spin-s tracking procedure

2.2.1 The general case

We begin by noticing that in general the generators of the Poincaré group are marked by external space-time (Lorentz) indices (see appendix A) and which we denote by small Greek letters μ,ν,λ,ρ, etc. next to representation specific indices (denoted by capital Latin letters $A,B,C,\ldots$ etc.) At times, like for example in ψ_μ, it may be possible and useful to separate the capital Latin letter indices into Lorentz and spinorial parts. Therefore, the most general form of the Pauli-Lubanski vector operator is given by

$$(\mathcal{W}_\lambda)_{AC} = \frac{1}{2}\epsilon_{\lambda\rho\sigma\mu}(M^{\rho\sigma})_{AC}p^\mu. \tag{15}$$

Its squared form is then written as

$$(\mathcal{W}_\lambda\mathcal{W}^\lambda)_{AB} = \frac{1}{4}\epsilon_{\lambda\rho\sigma\mu}(M^{\rho\sigma})_{AC}p^\mu\epsilon^\lambda{}_{\tau\xi\nu}(M^{\tau\xi})_{CB}p^\nu \equiv T_{AB\mu\nu}p^\mu p^\nu. \tag{16}$$

Notice that $T_{AB\mu\nu}$ is momentum independent.

Our pursued spin-tracking strategy will be the construction of covariant projectors onto the Poincaré invariant $\Psi^{(m,s)}$ sectors of the Lorentz representation of interest. Below, we illustrate this procedure for the simplest case of a generic Lorentz representation having only two Poincaré invariant subspaces with spins differing by one unit. We denote the maximal and minimal spins by s and $(s-1)$, respectively.

The covariant mass-m and spin-s tracking procedure can be outlined as follows. Construct the Poincaré covariant mass-m–spin-s, and mass-m–spin-$(s-1)$ projectors as

$$\mathcal{P}^{(m;s)}(p) = -\frac{1}{2s}\left(\frac{\mathcal{W}^2}{m^2} + s(s-1)\frac{p^2}{m^2}\mathbf{1}_{n\times n}\right), \tag{17}$$

$$\mathcal{P}^{(m;s-1)}(p) = \frac{1}{2s}\left(\frac{\mathcal{W}^2}{m^2} + s(s+1)\frac{p^2}{m^2}\mathbf{1}_{n\times n}\right), \tag{18}$$

where n stands for the dimensionality of the representation of interest. We must remark that these operators are projectors over well-defined spins whenever the particles are on mass shell. Indeed, using the basis of eigenstates of $\mathcal{W}^2$ it can be easily shown that on mass shell they satisfy the following relationships:

$$\begin{aligned}
\left[\mathcal{P}^{(m;s)}(p)\right]^2 &= \mathcal{P}^{(m;s)}(p),\\
\left[\mathcal{P}^{(m;s-1)}(p)\right]^2 &= \mathcal{P}^{(m;s-1)}(p),\\
\mathcal{P}^{(m;s)}(p)\mathcal{P}^{(m;s-1)}(p) &= 0,\\
\mathcal{P}^{(m;s)}(p) + \mathcal{P}^{(m;s-1)}(p) &= \mathbf{1}_{n\times n}.
\end{aligned} \tag{19}$$

The important point here is that in the general case, imposing the condition

$$\mathcal{P}^{(m;s)}(p)\Psi^{(m,s)} = \Psi^{(m,s)}, \tag{20}$$

will simultaneously track down the desired spin s, nullify spin $(s-1)$, and incorporate the mass shell condition

$$\mathcal{P}^{(m;s)}(p)\Psi^{(m,(s-1))} = 0, \tag{21}$$
$$(p^2 - m^2)\Psi^{(m,s)} = 0. \tag{22}$$

Thus our projectors simultaneously track down well-defined mass-m and well-defined spin-s or spin-$(s-1)$ eigenspaces. In its most general form, eq. (20) can be written as

$$\left[-\Gamma_{AB\mu\nu}p^\mu p^\nu + m^2\delta_{AB}\right]\Psi_B^{(m,s)} = 0, \tag{23}$$

where

$$\Gamma_{AB\mu\nu} = -\frac{1}{2s}\left(T_{AB\mu\nu} + s(s-1)\delta_{AB}\ g_{\mu\nu}\right), \tag{24}$$

with $T_{AB\mu\nu}$ defined in eq. (16) from above. Compared to ref. [34], the $\mathcal{P}^{(m;s)}(p)$ projectors contain the additional factor of p^2/m^2 in front of $s(s-1)\mathbf{1}_{n\times n}$ which is indispensable for fixing correctly the mass of the tracked state as visible through eq. (41) below.

2.2.2 The spin-1 case

The most important examples for applications of the covariant spin-tracking procedure are the four-vector, A_μ, and the four-vector spinor, ψ_μ. In the former case, using the explicit form for W^2 in eq. (A.6) from appendix A we obtain

$$\Gamma^P_{\alpha\beta\mu\nu} = g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}, \tag{25}$$

and eqs. (21), (23) yield just the Proca equation

$$\left[(-p^2+m^2)g_{\mu\nu} + p_\mu p_\nu\right] A^\nu = 0, \tag{26}$$

which can be derived from the following Lagrangian:

$$\begin{aligned}\mathcal{L}_P &= -\frac{1}{2}(\partial^\mu A^\alpha)\Gamma^P_{\alpha\beta\mu\nu}\partial^\nu A^\beta + \frac{m^2}{2}A^\alpha A_\alpha \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\alpha A_\alpha.\end{aligned} \tag{27}$$

It is quite instructive to rewrite eq. (26) in terms of the spin-1 and spin-0 projectors in A_μ, in turn denoted by $\mathbf{P}^{(1)}_{\mu\nu}$, and $\mathbf{P}^{(0)}_{\mu\nu}$, and defined as

$$\mathbf{P}^{(1)}_{\mu\nu} = \frac{(W^2)_{\mu\nu}}{-2p^2} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\,, \qquad \mathbf{P}^{(0)}_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}\,. \tag{28}$$

In so doing, one finds

$$\mathcal{P}^{(m,1)}_{\mu\nu}(p)A^\nu = \frac{p^2}{m^2}\mathbf{P}^{(1)}_{\mu\nu}A^\nu = A_\mu\,, \tag{29}$$

an equation which reveals the Poincaré invariant projector $\mathcal{P}^{(m,1)}(p)$ as the direct product of the mass-m and spin-1 projectors. The inverse to eq. (26) provides the Proca propagator as

$$\Pi^{\text{Proca}}_{\mu\nu} = \frac{\Delta^{\text{Proca}}_{\mu\nu}}{(p^2-m^2+i\varepsilon)}, \tag{30}$$

where

$$\Delta^{\text{Proca}}_{\mu\nu} = -g_{\mu\nu} + \frac{p^\mu p^\nu}{m^2} = -\mathbf{P}^{(1)}_{\mu\nu} + \frac{p^2-m^2}{m^2}\mathbf{P}^{(0)}_{\mu\nu}. \tag{31}$$

Throughout this paper, propagators are given in momentum space, hence momentum operators like $p^\mu \equiv i\partial^\mu$ in the projectors must be replaced by their eigenvalues. This simple example shows what one can anticipate from the application of the covariant mass-m and spin-$\frac{3}{2}$ tracking procedure to the four-vector spinor representation.

3 Free spin-$\frac{3}{2}$ beyond the Rarita-Schwinger framework

In this section we apply the spin-tracking procedure to the vector spinor representation. The decomposition of this space into Poincaré invariant sectors reads

$$\psi_\mu \longrightarrow [\Psi^{(m,\frac{3}{2})}]^{(2)} \oplus [\Psi^{(m,\frac{1}{2})}]^{(4)}, \tag{32}$$

where the subscript labels the multiplicity of the representation. Correspondingly, the Poincaré covariant spin-$\frac{3}{2}$ projector in eq. (17) becomes

$$\mathcal{P}^{(m;\frac{3}{2})}(p) = -\frac{1}{3}\left(\frac{\mathcal{W}^2}{m^2} + \frac{3}{4}\frac{p^2}{m^2}\mathbf{1}_{16\times16}\right). \tag{33}$$

As long as the projectors are per construction covariant, the equation of motion for spin-$\frac{3}{2}$ in ψ_μ and in any basis reads

$$\left[-\frac{1}{3}\left(\mathcal{W}^2 + \frac{3}{4}p^2\mathbf{1}_{16\times16}\right) - m^2\mathbf{1}_{16\times16}\right]\psi = 0. \tag{34}$$

In terms of the tensor $T_{AB\mu\nu}$, defined in eq. (16), the latter equation rewrites to

$$\left[-\frac{1}{3}T_{AB\mu\nu}p^\mu p^\nu - \left(\frac{1}{4}p^2 + m^2\right)\delta_{AB}\right]\psi^B = 0, \qquad A: \alpha\, a,\ B: \beta\, b, \tag{35}$$

where a is the spinorial index.

3.1 The $\mathcal{W}^2$- and p^2-driven spin-$\frac{3}{2}$ equations

In order to obtain the explicit form of eq. (35) in the interesting ψ_μ basis where Lorentz and spinor indices appear separated, we, first of all, have to find $T_{AB\mu\nu}$, a calculation that we present in appendix A. Insertion of eq. (A.14) from the appendix into eq. (35) amounts to the following free spin-$\frac{3}{2}$ wave equation:

$$[-K_{\alpha\beta} + m^2\ g_{\alpha\beta}]\psi^\beta = 0, \tag{36}$$

with

$$K_{\alpha\beta} \equiv \Gamma_{\alpha\beta\mu\nu}p^\mu p^\nu, \tag{37}$$

where, for the sake of simplicity, we suppressed the spinorial indices and defined $\Gamma_{\alpha\beta\mu\nu}$ as

$$\begin{aligned}\Gamma_{\alpha\beta\mu\nu} &= \frac{2}{3}\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \\ &\quad + \frac{1}{6}(\epsilon^\lambda{}_{\alpha\beta\mu}\gamma^5\sigma_{\lambda\nu} + \epsilon^\lambda{}_{\alpha\beta\nu}\gamma^5\sigma_{\lambda\mu}) \\ &\quad + \frac{1}{12}\sigma_{\lambda\mu}\sigma^\lambda{}_\nu g_{\alpha\beta} - \frac{1}{4}g_{\mu\nu}g_{\alpha\beta}\,.\end{aligned} \tag{38}$$

It immediately verifies that the operator $K_{\alpha\beta}$ satisfies the following relations:

$$\begin{aligned}p^\alpha K_{\alpha\beta} &= 0, \qquad K_{\alpha\beta}p^\beta = 0, \\ \gamma^\alpha K_{\alpha\beta} &= 0, \qquad K_{\alpha\beta}\gamma^\beta = 0.\end{aligned} \tag{39}$$

The resulting free particle equation reads

$$\Big[\left(-p^2+m^2\right) g_{\alpha\beta} + \frac{2}{3} p_\beta p_\alpha + \frac{1}{3}\left(p_\alpha\gamma_\beta + p_\beta\gamma_\alpha\right) \not{p} - \frac{1}{3}\gamma_\alpha \not{p}\gamma_\beta \not{p}\Big]\psi^\beta = 0. \tag{40}$$

It equivalently rewrites as

$$\mathcal{P}^{(m,\frac{3}{2})}_{\alpha\beta}(p)\psi^\beta = \frac{p^2}{m^2}\mathbf{P}^{(\frac{3}{2})}_{\alpha\beta}\,\psi^\beta = \psi_\alpha\,. \tag{41}$$

Here, $\mathbf{P}^{(\frac{3}{2})}_{\alpha\beta}$ stands for the spin-$\frac{3}{2}$ projector in ψ_μ and is given by

$$\mathbf{P}^{(\frac{3}{2})}_{\alpha\beta} = -\frac{1}{3}\left(\frac{\mathcal{W}^2_{\alpha\beta}}{p^2} + \frac{3}{4}g_{\alpha\beta}\right). \tag{42}$$

Equation (41) reveals the Poincaré invariant projector $\mathcal{P}^{(m,\frac{3}{2})}(p)$ as the direct product of a mass-m, and spin-$\frac{3}{2}$ projectors, much alike eq. (29) and as it should be. Using eqs. (39) allows to find that the four-vector spinor field satisfies

$$[p^2-m^2]\psi_\alpha = 0, \tag{43}$$

$$\gamma_\alpha\psi^\alpha = 0, \tag{44}$$

$$p_\alpha\psi^\alpha = 0. \tag{45}$$

3.2 The spin-$\frac{3}{2}$ Lagrangian beyond Rarita-Schwinger

The equation of motion (36) can be derived from the following manifestly Hermitian Lagrangian:

$$\mathcal{L}_{\rm free} = -\frac{1}{2}[(\partial^\mu\overline{\psi}^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu\psi^\beta + (\partial^\nu\overline{\psi}^\beta)\overline{\Gamma}_{\alpha\beta\mu\nu}\partial^\mu\psi^\alpha] + m^2\overline{\psi}^\alpha\psi_\alpha\,, \tag{46}$$

where

$$\overline{\Gamma}_{\alpha\beta\mu\nu} \equiv \gamma^0(\Gamma_{\alpha\beta\mu\nu})^\dagger\gamma^0. \tag{47}$$

Using eq. (A.14) it is easy to show that

$$\overline{\Gamma}_{\alpha\beta\mu\nu} = \Gamma_{\beta\alpha\nu\mu}, \tag{48}$$

hence our Lagrangian can be rewritten to the simpler form

$$\mathcal{L}_{\rm free} = -(\partial^\mu\overline{\psi}^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu\psi^\beta + m^2\overline{\psi}^\alpha\psi_\alpha. \tag{49}$$

Subjecting eq. (36) and its adjoint to standard algebraic manipulations, or calculating directly the Noether current for the usual phase invariance of the Lagrangian (49) we obtain

$$j_\mu = (i\partial^\nu\overline{\psi}^\alpha)\Gamma_{\alpha\beta\nu\mu}\psi^\beta - \overline{\psi}^\alpha\Gamma_{\alpha\beta\mu\nu}i\partial^\nu\psi^\beta\,, \tag{50}$$

as a conserved current

$$\partial^\mu j_\mu = 0. \tag{51}$$

3.3 The spin-$\frac{3}{2}$ propagator

The formal calculation of the two-point Green function in our theory requires to work out the quantization of the formalism which is presently under investigation and beyond the scope of this paper. However, we calculated the propagator as the inverse of the operator $(-K_{\alpha\beta}+m^2g_{\alpha\beta})$. In so doing, we obtain

$$\Pi_{\alpha\beta} = \frac{\Delta_{\alpha\beta}}{(p^2-m^2+i\varepsilon)}\,, \tag{52}$$

where

$$\Delta_{\alpha\beta} = -g_{\alpha\beta} + \frac{2}{3m^2}p_\beta p_\alpha + \frac{1}{3m^2}\left(p_\alpha\gamma_\beta+p_\beta\gamma_\alpha\right)\not{p} - \frac{1}{3m^2}\gamma_\alpha \not{p}\gamma_\beta \not{p}\,. \tag{53}$$

It is instructive to rewrite this tensor in terms of the projectors over well-defined spins. The result is

$$\Delta_{\alpha\beta} = -\mathbf{P}^{(\frac{3}{2})}_{\alpha\beta} + \frac{p^2-m^2}{m^2}\mathbf{P}^{(\frac{1}{2})}_{\alpha\beta}\,, \tag{54}$$

with

$$\mathbf{P}^{(\frac{1}{2})}_{\alpha\beta} = \frac{\mathcal{W}^2_{\alpha\beta}}{3p^2} + \frac{5}{4}g_{\alpha\beta}, \tag{55}$$

being the projector on spin-$\frac{1}{2}$ in ψ_μ. Equation (54) shows that off-shell the four-vector spinor carries all its lower-spin components, much alike the case of the four-vector in the description of "off-shell" electroweak gauge bosons [19].

That $\Pi_{\alpha\beta}$ is the inverse to the free particle equation can be easily shown using eq. (41) in combination with the nilpotent and orthogonality properties of the projectors. The similarity of the spin-$\frac{3}{2}$ propagator in eq. (54) and the A_μ-propagator in eq. (31) can hardly be overlooked.

Finally, the $\Delta_{\alpha\beta}$ operators have the following simple properties:

$$\begin{aligned} p^\alpha\Delta_{\alpha\beta} &= \frac{1}{m^2}\left(p^2-m^2\right)p_\beta,\\ \Delta_{\alpha\beta}p^\beta &= \frac{1}{m^2}\left(p^2-m^2\right)p_\alpha\,,\\ \gamma^\alpha\Delta_{\alpha\beta} &= \frac{1}{m^2}\left(p^2-m^2\right)\gamma_\beta,\\ \Delta_{\alpha\beta}\gamma^\beta &= \frac{1}{m^2}\left(p^2-m^2\right)\gamma_\alpha. \end{aligned} \tag{56}$$

Before concluding the current section, we wish to notice that eq. (41) finds a marginal mentioning in ref. [35], however without any discussion on its link to $\mathcal{W}^2$ and without exploiting its potential in the description of spin-$\frac{3}{2}$.

4 Symmetries of the Lagrangian

This section is devoted to the symmetries of eq. (36) in the massless limit, their impact on the massive case and its relation to the prime question of the uniqueness of the Lagrangian in eq. (49).

4.1 Parameter independence of the massless case

In the massless case eq. (36) remains invariant under the following "gauge" transformation:

$$\psi^\beta \to \psi'^\beta = \psi^\beta + p^\beta \chi, \tag{57}$$

with χ being an arbitrary spinor. This invariance appears as a consequence of

$$K_{\alpha\beta} p^\beta = 0. \tag{58}$$

It is the same as the conventional "gauge" symmetry satisfied also by the $(A = -1)$ version of the massless RS equation of motion in eq. (7) which has been extensively used in particular in supergravity [35]. More recently, the symmetry in eq. (57) has been exploited as a guiding principle in the construction of chiral Lagrangians for light baryons [36]. The relation

$$K_{\alpha\beta}\gamma^\beta = 0, \tag{59}$$

in eq. (39) implies invariance of eq. (36) under the point transformation

$$\psi^\beta \to \psi'^\beta = \psi^\beta + \gamma^\beta \chi. \tag{60}$$

At that stage, the question on the uniqueness of the formalism proposed here comes up. In order to answer this question, let us first perform the following "rotation" within the unphysical spin-$\frac{1}{2}$ sector

$$\psi^\beta \to \psi'^\beta = R^\beta{}_\rho \left(\frac{A}{2}\right) \psi^\rho, \tag{61}$$

with $R(w)$ given by eq. (10). In so doing, one produces the A-dependent Lagrangian

$$\begin{aligned} \mathcal{L}(A) &= -(\partial^\mu \overline{\psi'}^\alpha) \Gamma_{\alpha\beta\mu\nu} \partial^\nu \psi'^\beta \\ &= -\left(\partial^\mu \overline{\psi}^\sigma\right) \Gamma_{\sigma\rho\mu\nu}(A) \partial^\nu \psi^\rho, \end{aligned} \tag{62}$$

with

$$\Gamma_{\sigma\rho\mu\nu}(A) = R_\sigma{}^\alpha \left(\frac{A}{2}\right) \Gamma_{\alpha\beta\mu\nu} R^\beta{}_\rho \left(\frac{A}{2}\right). \tag{63}$$

At first glance, the equation of motion resulting from the latter Lagrangian presents itself A-dependent as

$$-\Gamma_{\sigma\rho\mu\nu}(A) p^\mu p^\nu \psi^\rho = 0. \tag{64}$$

However, this impression is misleading. Indeed, by making use of eq. (59) allows to eliminate the A-dependence according to

$$\begin{aligned} K_{\sigma\rho}(A) &\equiv \Gamma_{\sigma\rho\mu\nu}(A) p^\mu p^\nu \\ &= R_\sigma{}^\alpha \left(\frac{A}{2}\right) K_{\alpha\beta} R^\beta{}_\rho \left(\frac{A}{2}\right) = K_{\sigma\rho}. \end{aligned} \tag{65}$$

This means that in the massless limit our eq. (36) is unique. This uniqueness is of course related to the invariance under the point transformations in eq. (60) as can be easily seen in choosing the χ spinor as $\chi = \frac{A}{2}\gamma \cdot \psi$. The only way for two different Lagrangians to produce one and the same equation of motion is that they differ by a total divergence term according to

$$\mathcal{L}(A) = \mathcal{L} - \partial^\mu \Lambda_\mu(A). \tag{66}$$

That this is indeed the case follows directly from the explicit calculation of $\Lambda_\mu(A)$ giving

$$\begin{aligned} \Lambda_\mu(A) = \frac{A}{2} &\left[\overline{\psi}^\sigma \gamma_\sigma \gamma^\alpha \Gamma_{\alpha\rho\mu\nu} \partial^\nu \psi^\rho + (\partial^\tau \overline{\psi}^\sigma) \Gamma_{\sigma\beta\tau\mu} \gamma^\beta \gamma_\rho \psi^\rho \right] \\ &+ \frac{A^2}{4} \overline{\psi}^\sigma \gamma_\sigma \gamma^\alpha \Gamma_{\alpha\beta\mu\nu} \gamma^\beta \gamma_\rho \partial^\nu \psi^\rho. \end{aligned} \tag{67}$$

In this manner the parameter independence of our suggested formalism in the massless case establishes neatly. Notice however that the operator $K_{\alpha\beta}$ is not invertible, meaning that the propagator in eq. (52) is singular in the massless case. Same occurs for spin-1 due to the singularity of the massless operator in eq. (26). This problem reflects the gauge freedom of massless theories and is resolved by introducing a gauge fixing term into the Lagrangian, a technique that will acquire importance in the following.

4.2 Extrapolation to the massive case

The mass term in eq. (36) breaks both the gauge symmetry and the invariance under point transformations defined in eqs. (59), (60). In the massive case calculations similar to those presented in the previous subsection, yield the following genuinely A-dependent Lagrangian

$$\mathcal{L}(A) = \mathcal{L}_K + \overline{\psi}^\sigma \left[M^2(A)\right]_{\sigma\rho} \psi^\rho - \partial^\mu \Lambda_\mu(A), \tag{68}$$

with

$$\begin{aligned} \left[M^2(A)\right]_{\sigma\rho} &= m^2 R_\sigma{}^\alpha \left(\frac{A}{2}\right) R_{\alpha\rho} \left(\frac{A}{2}\right) \\ &= m^2 R_{\sigma\rho}(A(1+A)). \end{aligned} \tag{69}$$

In order to understand the effect of the "rotation" within the spin-$\frac{1}{2}$ sector, let us rewrite the equation of motion in terms of the projectors as

$$\left[-p^2 \left(\mathbf{P}^{(\frac{3}{2})}\right)_{\mu\nu} + m^2 g_{\mu\nu}\right] \psi^\nu = 0. \tag{70}$$

Next, we shall separate the spin-$\frac{3}{2}$ from the spin-$\frac{1}{2}$ mass term as

$$\left[\left(-p^2 + m^2\right) \left(\mathbf{P}^{(\frac{3}{2})}\right)_{\mu\nu} + m^2 (\mathbf{P}^{(\frac{1}{2})})_{\mu\nu}\right] \psi^\nu = 0. \tag{71}$$

The A-dependent equation of motion can be written as

$$R\left(\frac{A}{2}\right) \left[\left(-p^2 + m^2\right) \mathbf{P}^{(\frac{3}{2})} + m^2 \mathbf{P}^{(\frac{1}{2})}\right] R\left(\frac{A}{2}\right) \psi = 0. \tag{72}$$

It is convenient now to use the $\mathbf{P}^{(\frac{3}{2})}$, $\mathbf{P}_{11}^{(\frac{1}{2})}$, and $\mathbf{P}_{22}^{(\frac{1}{2})}$ projectors, and the so-called "switch" operators, $\mathbf{P}_{12}^{(\frac{1}{2})}, \mathbf{P}_{12}^{(\frac{1}{2})}$ which can be found, among others, in ref. [35] and read

$$\begin{aligned}
\left(\mathbf{P}^{(\frac{3}{2})}\right)_{\mu\nu} &= g_{\mu\nu} - \frac{1}{3}\gamma_\mu\gamma_\nu - \frac{1}{3p^2}(\not p\gamma_\mu p_\nu + p_\mu\gamma_\nu \not p), \\
\left(\mathbf{P}_{11}^{(\frac{1}{2})}\right)_{\mu\nu} &= -\frac{p_\mu p_\nu}{p^2} + \frac{1}{3}\gamma_\mu\gamma_\nu + \frac{1}{3p^2}(\not p\gamma_\mu p_\nu + p_\mu\gamma_\nu \not p), \\
\left(\mathbf{P}_{22}^{(\frac{1}{2})}\right)_{\mu\nu} &= \frac{p_\mu p_\nu}{p^2}, \\
\left(\mathbf{P}_{12}^{(\frac{1}{2})}\right)_{\mu\nu} &= \frac{1}{\sqrt{3}p^2}\left(p_\mu p_\nu - \not p\gamma_\mu p_\nu\right), \\
\left(\mathbf{P}_{21}^{(\frac{1}{2})}\right)_{\mu\nu} &= \frac{1}{\sqrt{3}p^2}\left(-p_\mu p_\nu + \not p p_\mu\gamma_\nu\right).
\end{aligned} \tag{73}$$

The above operators constitute a complete set in the vector spinor representation space and satisfy the following orthogonality and completeness relations:

$$\begin{aligned}
\mathbf{P}_{ij}^a\mathbf{P}_{kl}^b &= \delta^{ab}\delta_{jk}\mathbf{P}_{il}^b, \\
\mathbf{P}^{(\frac{3}{2})} + \mathbf{P}_{11}^{(\frac{1}{2})} + \mathbf{P}_{22}^{(\frac{1}{2})} &= 1.
\end{aligned} \tag{74}$$

Further useful relations are

$$\begin{aligned}
\not p\mathbf{P}^{(\frac{3}{2})} &= \mathbf{P}^{(\frac{3}{2})}\not p, \\
\not p\mathbf{P}_{ij}^{(\frac{1}{2})} &= \pm\mathbf{P}_{ij}^{(\frac{1}{2})}\not p, \qquad \begin{array}{l} + \text{ if } i=j, \\ - \text{ if } i\neq j, \end{array} \\
\gamma^\mu\mathbf{P}_{\mu\nu}^{(\frac{3}{2})} &= \mathbf{P}_{\mu\nu}^{(\frac{3}{2})}\gamma^\nu = p^\mu\mathbf{P}_{\mu\nu}^{(\frac{3}{2})} = \mathbf{P}_{\mu\nu}^{(\frac{3}{2})}p^\nu = 0, \\
\gamma^\mu\left(\mathbf{P}^{(\frac{1}{2})}\right)_{\mu\nu} &= \gamma_\nu, \qquad \left(\mathbf{P}^{(\frac{1}{2})}\right)_{\mu\nu}\gamma^\nu = \gamma_\mu, \\
p^\mu\left(\mathbf{P}^{(\frac{1}{2})}\right)_{\mu\nu} &= p_\nu, \qquad \left(\mathbf{P}^{(\frac{1}{2})}\right)_{\mu\nu}p^\nu = p_\mu.
\end{aligned} \tag{75}$$

The $\mathbf{P}^{(\frac{3}{2})}$ projector was related to the squared Pauli-Lubanski vector in eq. (42) whereas the projector in eq. (55) expresses as

$$\mathbf{P}^{(\frac{1}{2})} = \mathbf{P}_{11}^{(\frac{1}{2})} + \mathbf{P}_{22}^{(\frac{1}{2})}. \tag{76}$$

These relations can be exploited to cast eq. (72) into the form which manifestly shows that *solely the mass term in the spin-$\frac{1}{2}$ sector is affected by the point transformation*,

$$\left[\left(-p^2 + m^2\right)\mathbf{P}^{(\frac{3}{2})} + m^2 R\left(\frac{A}{2}\right)\mathbf{P}^{(\frac{1}{2})}R\left(\frac{A}{2}\right)\right]\psi = 0. \tag{77}$$

Under the above "rotation" the mass matrix changes from a diagonal form in both spin-sectors to one that remains diagonal only in the spin-$\frac{3}{2}$ but becomes non-diagonal in the spin-$\frac{1}{2}$ sector. This non-diagonality is irrelevant for on-shell particles as is well visible upon contracting eq. (77) with γ^σ and p^σ, and recalling that the invertibility of $R(\frac{A}{2})$ requires $A \neq -\frac{1}{2}$,

$$\begin{aligned}
m^2(1+2A)^2\gamma\cdot\psi = 0, &\Rightarrow \gamma\cdot\psi = 0, \\
m^2(p\cdot\psi + A(A+1)\gamma\cdot\psi) = 0, &\Rightarrow \partial\cdot\psi = 0.
\end{aligned} \tag{78}$$

In this fashion, the form of the mass matrix in the spin-$\frac{1}{2}$ sector in the free equation remains without importance. However, it becomes relevant for the off-mass shell propagator. In order to see this, notice, that the A-dependent propagator is easily calculated in the following way. Let us first denote $(-K_{\alpha\beta} + m^2 g_{\alpha\beta})$ in eq. (36) by $\mathcal{O}_{\alpha\beta}$. Upon the $R(\frac{A}{2})$ transformation, $\mathcal{O}_{\alpha\beta}$ becomes $\mathcal{O}(A) = R(\frac{A}{2})\mathcal{O}R(\frac{A}{2})$ and as long as the new propagator comes from $\Pi(A)\mathcal{O}(A) = 1$ then one finds $\Pi(A)$ from

$$R\left(\frac{A}{2}\right)^{-1}\Pi R\left(\frac{A}{2}\right)^{-1}R\left(\frac{A}{2}\right)\mathcal{O}R\left(\frac{A}{2}\right) = 1. \tag{79}$$

The latter equation leads to the following A-dependent massive propagator:

$$\Pi(A) = \frac{-\mathbf{P}^{(\frac{3}{2})} + \frac{p^2-m^2}{m^2}R^{-1}\left(\frac{A}{2}\right)\mathbf{P}^{(\frac{1}{2})}R^{-1}\left(\frac{A}{2}\right)}{p^2 - m^2 + i\epsilon}. \tag{80}$$

Therefore, the remnant A-dependence affects only the spin-$\frac{1}{2}$ contribution to the off-shell propagator. In other words, the parameter dependence of the massive spin-$\frac{3}{2}$ propagator appeared as a consequence of respecting in the massive theory the symmetries of the massless case. The situation is by no means new. In a similar way, the gauge symmetry of the massless spin-1 theory is respected by the massive one on the cost of a parameter-dependent spin-0 sector in the massive gauge boson propagator, a subject that we treat in some detail the following subsection.

4.3 Off-shell propagators and parameter dependence

In this subsection we shall make the case that the massive off-shell spin-$\frac{3}{2}$ propagator proposed here is of the type of the propagators which appear in massive gauge theories and that its parameter dependence reflects the symmetries of the massless theory, one of them being the gauge freedom. In order to see this we begin with casting the essentials of the standard massive gauge theories in the language systematically used by us through this paper, namely the one of the covariant projectors as applied to the spin-0, and spin-1 sectors in the $(\frac{1}{2},\frac{1}{2})$-space. Then, we analyze the parameter dependence of our propagator in the light of the symmetries of the massless Lagrangian.

4.3.1 Gauge fixing in the massive spin-1 propagator

The problem of the parameter dependence of the off-shell propagators is quite general indeed and appears in massive spin-1 gauge theories. In the massless case, eq. (26) is not invertible as visible from eqs. (30), (31). The non-invertibility reflects the gauge freedom and it is circumvented by the introduction of appropriate gauge fixing terms into the Lagrangian according to

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2a}\left(\partial^\mu A_\mu\right)^2. \tag{81}$$

Here, the J^μ current depends on the matter fields. The wave equation associated with the latter Lagrangian reads

$$\left[-p^2 g_{\mu\nu} - \left(\frac{1}{a}-1\right) p_\mu p_\nu\right] A^\nu = J_\mu. \tag{82}$$

Equation (82) is now invertible and leads to the following a-dependent propagator:

$$\begin{aligned}\Pi_{\mu\nu}(a) &= \frac{1}{p^2+i\varepsilon}\left[-g_{\mu\nu} + (1-a)\frac{p_\mu p_\nu}{p^2}\right]\\ &= \frac{1}{p^2+i\varepsilon}\left[-\mathbf{P}^{(1)}_{\mu\nu} - a\mathbf{P}^{(0)}_{\mu\nu}\right],\end{aligned} \tag{83}$$

with the spin-1 projectors from eq. (28). Choosing specific values for a is standard and known as "gauge fixing". The $a = 1$ value in eq. (83) is known as Feynman's gauge while $a = 0$ gives the Landau propagator.

The link to the massive case is established by noticing that the gauge condition becomes a constraint that is preserved under interactions whenever the massive gauge boson is coupled to a conserved current. This is possible only within the context of mass generation via the Higgs mechanism, a possibility which we highlight in brief in what follows. To be specific, in massive gauge theories one faces the problem to guarantee validity of the gauge condition $\partial \cdot A = 0$. For this purpose and in analogy to the massless theory one introduces a Lagrange multiplier into the Lagrangian according to

$$\mathcal{L}_P = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu - \frac{1}{2a}\left(\partial^\mu A_\mu\right)^2 - J^\mu A_\mu. \tag{84}$$

The resulting massive equation of motion now becomes

$$\left[\left(-p^2+m^2\right)g_{\mu\nu} - \left(\frac{1}{a}-1\right)p_\mu p_\nu\right]A^\nu = J_\mu, \tag{85}$$

or equivalently,

$$\left[\left(-p^2+m^2\right)\mathbf{P}^{(1)}_{\mu\nu} - \frac{1}{a}\left(p^2 - am^2\right)\mathbf{P}^{(0)}_{\mu\nu}\right]A^\nu = J_\mu. \tag{86}$$

Here, p_μ denotes the four-momentum of the gauge boson. The associated propagator is well known and obtained as

$$\begin{aligned}\Pi_{\mu\nu}(a) &= \frac{1}{p^2-m^2+i\varepsilon}\left[-\mathbf{P}^{(1)}_{\mu\nu} - a\frac{p^2-m^2}{p^2-am^2}\mathbf{P}^{(0)}_{\mu\nu}\right]\\ &= \frac{-g_{\mu\nu} + (1-a)\frac{p_\mu p_\nu}{p^2-am^2}}{p^2-m^2+i\varepsilon}.\end{aligned} \tag{87}$$

The Proca propagator in eq. (30) corresponds to the particular choice of $a = \infty$, and appears singular in the massless case, much alike our propagator in eq. (52).

Although not obvious, the latter expression is related to the conventional mass generation mechanism for gauge bosons via the Higgs mechanism [37]. In order to illustrate this statement one couples the gauge boson to a charged scalar ("Higgs") field defined as

$$\phi = \frac{1}{\sqrt{2}}(v + \chi_1 + i\chi_2), \tag{88}$$

and obtains the equation of motion from

$$\begin{aligned}\partial^\mu F_{\mu\nu} = j_\nu &= e\left[\phi^*(i\partial_\nu\phi) - (i\partial_\nu\phi^*)\phi\right] - 2e^2 A_\nu\phi^*\phi\\ &= -m^2 A_\nu - m\partial_\nu\chi_2 - e\left[\chi_1\partial_\nu\chi_2 - \chi_2\partial_\nu\chi_1\right]\\ &\quad -e^2 A_\nu(\chi_1^2 + 2v\chi_1 + \chi_2^2),\end{aligned} \tag{89}$$

with $m \equiv ev$. In considering now the special gauge

$$\partial_\mu A^\mu = m\xi\chi_2, \tag{90}$$

where ξ is an arbitrary parameter one is led to

$$\chi_2 = \frac{1}{m\xi}\partial_\mu A^\mu. \tag{91}$$

With that the equation of motion for the gauge boson becomes

$$\begin{aligned}\left[-p^2+m^2\right]A_\nu - \left(\frac{1}{\xi}-1\right)p_\nu p^\mu A_\mu =&\\ ie\left[\chi_1 p_\nu\chi_2 - \chi_2 p_\nu\chi_1\right]&\\ -e^2A_\nu(\chi_1^2 + 2v\chi_1 + \chi_2^2).&\end{aligned} \tag{92}$$

The right-hand side of the latter equation contains interactions of the gauge boson field with the Higgs field, χ_1, as well as self-interactions. Its left-hand side can be inverted to yield the well-known 't Hooft propagator

$$\begin{aligned}\Pi^{\text{'t Hooft}}_{\mu\nu}(\xi) &= \frac{-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2-\xi m^2}}{p^2-m^2+i\varepsilon}\\ &= \frac{1}{p^2-m^2+i\varepsilon}\left[-\mathbf{P}^{(1)}_{\mu\nu} - \xi\frac{p^2-m^2}{p^2-\xi m^2}\mathbf{P}^{(0)}_{\mu\nu}\right].\end{aligned} \tag{93}$$

The 't Hooft propagator describes massive vector particles whose mass has been generated via the Higgs mechanism. Now one recovers the propagator in eq. (87) in assuming $v \neq 0$, and $\xi = a$ in eqs. (88), (90). The $v = 0$ value implies $m = 0$ and the absence of spontaneous symmetry breaking. The resulting massless propagator coincides with the one given in eq. (83). This brief reminiscence of the massive spin-1 case is suggestive of the idea to view the A-dependence of the massive spin-$\frac{3}{2}$ propagator in eq. (80) in the light of gauge fixing, an idea that we execute in the next subsection.

4.3.2 "Gauge" fixing in the spin-$\frac{3}{2}$ theory

In parallel to the spin-1 description, we here shall term to the freedom in the choice of the massless ψ^β provided by the symmetries in eqs. (57), and (60) as "gauge" freedom (better, "gauge" freedoms), and fix it by including into the Lagrangian the associated Lagrange multipliers according to

$$\begin{aligned}\mathcal{L} = &-(\partial^\mu\overline{\psi}^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu\psi^\beta - \frac{1}{a}\left(\partial^\mu\overline{\psi}_\mu\right)\left(\partial^\alpha\psi_\alpha\right)\\ &-\frac{\mu^2}{b}\left(\overline{\psi}_\mu\gamma^\mu\right)\left(\gamma^\alpha\psi_\alpha\right) - \overline{\psi}^\mu f_\mu - \overline{f}^\mu\psi_\mu.\end{aligned} \tag{94}$$

Notice that we use the Lagrange multiplier μ^2/b where μ is an arbitrary (but fixed) mass scale which allows to treat the parameters as dimensionless. This new Lagrangian yields now the equation of motion as

$$\left[-K_{\alpha\beta} - \frac{1}{a}p_\alpha p_\beta - \frac{\mu^2}{b}\gamma_\alpha\gamma_\beta\right]\psi^\beta = f_\alpha, \qquad (95)$$

where f_μ is some fermion current involving other fields. The operator on the left-hand side of the latter equation is now invertible. In terms of the projectors in eq. (73) it is given by

$$\mathcal{O}^{(m=0)}(a,b) = -p^2\mathbf{P}^{(\frac{3}{2})} - \frac{3\mu^2}{b}\mathbf{P}^{(\frac{1}{2})}_{11} - \frac{1}{ab}(bp^2 + a\mu^2)\mathbf{P}^{1/2}_{22} - \frac{\sqrt{3}\mu^2}{b}\left(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21}\right), \qquad (96)$$

where we used

$$\gamma_\alpha\gamma_\beta = \left[3\mathbf{P}^{(\frac{1}{2})}_{11} + \mathbf{P}^{(\frac{1}{2})}_{22} + \sqrt{3}\left(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21}\right)\right]_{\alpha\beta}. \qquad (97)$$

The propagator in the "(a,b)-gauge" is now found to be

$$\Pi^{(m=0)}(a,b) = \frac{\Delta^{(m=0)}(a,b)}{p^2 + i\varepsilon}. \qquad (98)$$

Here,

$$\Delta^{(m=0)}(a,b) = -\mathbf{P}^{(\frac{3}{2})} - \frac{1}{3}\left[\left(\frac{b}{\mu^2}p^2 + a\right)\mathbf{P}^{(\frac{1}{2})}_{11} + 3a\mathbf{P}^{(\frac{1}{2})}_{22} + \sqrt{3}a\left(\mathbf{P}^{(\frac{1}{2})}_{12} - \mathbf{P}^{(\frac{1}{2})}_{21}\right)\right]. \qquad (99)$$

This is the spin-$\frac{3}{2}$ analogous to the massless spin-1 propagator in eq. (83).

Next, we extrapolate to the massive case. Adding the mass term to the Lagrangian results in

$$\mathcal{L} = -(\partial^\mu\overline{\psi}^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu\psi^\beta + m^2\overline{\psi}^\alpha\psi_\alpha - \frac{1}{a}\left(\partial^\mu\overline{\psi}_\mu\right)\left(\partial^\alpha\psi_\alpha\right) - \frac{\mu^2}{b}\left(\overline{\psi}_\mu\gamma^\mu\right)\left(\gamma^\alpha\psi_\alpha\right) - \overline{\psi}^\mu f_\mu - \overline{f}^\mu\psi_\mu. \qquad (100)$$

The massive equation of motion reads

$$\left[-K_{\alpha\beta} + m^2 g_{\alpha\beta} - \frac{1}{a}p_\alpha p_\beta - \frac{\mu^2}{b}\gamma_\alpha\gamma_\beta\right]\psi^\beta = f_\alpha. \qquad (101)$$

In the $a=\infty$-"gauge" eq. (101) corresponds to the "rotated" eq. (77), modulo the identification $\frac{\mu^2}{b} = -A(1+A)m^2$. *This observation reveals the effect of the rotation in eq.* (77) *just as a change in the "gauge" used in the massless case (massive case, in reference to the Higgs mechanism).*

In terms of the projectors in eq. (73), the operator acting on the field on the left-hand side in eq. (101) reads

$$\mathcal{O}(a,b) = \left(-p^2 + m^2\right)\mathbf{P}^{(\frac{3}{2})} - \frac{1}{b}(3\mu^2 - bm^2)\mathbf{P}^{(\frac{1}{2})}_{11} - \frac{1}{ab}(bp^2 + a(\mu^2 - bm^2))\mathbf{P}^{1/2}_{22} - \frac{\sqrt{3}\mu^2}{b}\left(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21}\right). \qquad (102)$$

This operator has an inverse which is calculated as

$$\Pi(a,b) = \frac{\Delta(a,b)}{p^2 - m^2 + i\varepsilon}, \qquad (103)$$

with

$$\Delta(a,b) = -\mathbf{P}^{(\frac{3}{2})} - b\frac{p^2 - m^2}{(3\mu^2 - bm^2)(bp^2 + a(\mu^2 - bm^2)) - 3a\mu^4} \times\left[\left(bp^2 + a(\mu^2 - bm^2)\right)\mathbf{P}^{(\frac{1}{2})}_{11} + a\left(3\mu^2 - bm^2\right)\mathbf{P}^{(\frac{1}{2})}_{22} - \sqrt{3}a\mu^2(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21})\right]. \qquad (104)$$

The similarity of the massive spin-$\frac{3}{2}$ off-shell propagator in eq. (103) with the nominator from eq. (104) to the t'Hooft propagator in eq. (93) is hardly to be overlooked. In both cases the parameter dependence invokes propagation of the unphysical spin-sectors that have been excluded on-shell. This observation is suggestive of the idea to handle the parameter dependence in eq. (104) in the spirit of gauge fixing in massive theories.

In the $b \to \infty$ limit one finds

$$\Pi(a,\infty) = \frac{-\mathbf{P}^{(\frac{3}{2})} + \frac{p^2-m^2}{m^2}\mathbf{P}^{(\frac{1}{2})}_{11} - a\frac{p^2-m^2}{p^2-am^2}\mathbf{P}^{(\frac{1}{2})}_{22}}{p^2 - m^2 + i\varepsilon}, \qquad (105)$$

whereas for $a \to \infty$ one obtains the propagator that takes into account the $\gamma\cdot\psi = 0$ constraint alone,

$$\Pi(\infty,b) = \frac{\Delta(\infty,b)}{p^2 - m^2 + i\varepsilon}. \qquad (106)$$

Here,

$$\Delta(\infty,b) = -\mathbf{P}^{(\frac{3}{2})} - b\frac{p^2 - m^2}{(3\mu^2 - bm^2)(\mu^2 - bm^2) - 3\mu^4} \times\left[\left(\mu^2 - bm^2\right)\mathbf{P}^{(\frac{1}{2})}_{11} + \left(3\mu^2 - bm^2\right)\mathbf{P}^{(\frac{1}{2})}_{22} - \sqrt{3}\mu^2(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21})\right]. \qquad (107)$$

Notice that neither $\Pi(a,\infty)$, nor $\Pi(\infty,b)$ are free from singularities in the massless limit. Nonetheless, the general propagator in eq. (103) that incorporates both symmetries of the massless theory is *not singular* for $m=0$ in which case one recovers the propagator in eq. (98).

In the massive case, the simplest choice for the mass scale would be $\mu^2 = m^2$ in which case the general propagator is given by eq. (103) with the $\Delta(a,b)$ operator defined in eq. (104) being replaced by

$$\Delta(a,b) = -\mathbf{P}^{(\frac{3}{2})} - \frac{b}{m^2}\frac{p^2 - m^2}{(3-b)(bp^2 + a(1-b)m^2) - 3am^2} \times\left[\left(bp^2 + a(1-b)m^2\right)\mathbf{P}^{(\frac{1}{2})}_{11} + a(3-b)m^2\mathbf{P}^{(\frac{1}{2})}_{22} - \sqrt{3}am^2\left(\mathbf{P}^{(\frac{1}{2})}_{12} + \mathbf{P}^{(\frac{1}{2})}_{21}\right)\right]. \qquad (108)$$

Obviously, this expression is not suited for taking the $m \to 0$ limit. Finally, the counterpart to the spin-1 Landau propagator is obtained for $a = b = 0$ in which case only spin-$\frac{3}{2}$ is propagated,

$$\Pi(0,0) = \frac{-\mathbf{P}^{(\frac{3}{2})}}{p^2 - m^2 + i\varepsilon} \,. \tag{109}$$

Summarizing this section, in the massless case our equation of motion is unique and has as two important symmetries: i) the invariance under the gauge transformations in eq. (57), and ii) the invariance under the point transformations in eq. (60). As long as Lagrangians differing by "rotations" within the spin-$\frac{1}{2}$ sector are equivalent, the massless formalism is unique. Mass terms break the above symmetries in such a way that the $\gamma \cdot \psi = 0$, and $\partial \cdot \psi = 0$ conditions (occasionally termed to as "gauge" conditions) evolve to constraints. When properly taken into account, the symmetries related to these constraints yield a family of propagators whose spin-$\frac{1}{2}$ sectors depend on two parameters (termed to by us as "gauge" parameters in reference to the associated symmetries in the massless case). The propagator in eq. (52) represents just one of the members of this family. In analogy to massive gauge theories, the parameter-dependent terms in our off-shell propagator can be thought of as terms associated with "gauge fixing". Alternatively, the mass terms may be generated via the Higgs mechanism, an interesting possibility presently under investigation. From that perspective, the formalism presented here seems to be a good candidate for the description of massive spin-$\frac{3}{2}$ gauge fields. Before closing this section we would like to remark that in the conventional Rarita-Schwinger formalism it is not possible to interpret the A-dependence within the context of gauge fixing because the invariance under the transformation in eq. (7) is not general but an exclusive privilege of the $A = -1$ case in eq. (57), and the symmetry in eq. (60) is even completely absent.

5 Interacting spin-$\frac{3}{2}$ particles

The interacting theory is now obtained in the standard way in gauging the Lagrangian in eq. (46) with the result

$$\mathcal{L} = -(D^{\dagger\mu}\overline{\psi}^{\alpha})\Gamma_{\alpha\beta\mu\nu}D^{\nu}\psi^{\beta} + m^2\overline{\psi}^{\alpha}\psi_{\alpha}, \tag{110}$$

where $D^{\mu} = \partial^{\mu} - ieA^{\mu}$ is the covariant derivative, $(-e)$ denotes the charge of the particle.

This Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}, \tag{111}$$

with

$$\begin{aligned}\mathcal{L}_{\text{int}} &= ie\left[(\partial^{\nu}\overline{\psi}^{\alpha})\Gamma_{\alpha\beta\nu\mu}\psi^{\beta} - \overline{\psi}^{\alpha}\Gamma_{\alpha\beta\mu\nu}\partial^{\nu}\psi^{\beta}\right]A^{\mu}\\ &\quad -e^2\overline{\psi}^{\alpha}\Gamma_{\alpha\beta\mu\nu}\psi^{\beta}A^{\mu}A^{\nu}\\ &= ej_{\mu}A^{\mu} - e^2\overline{\psi}^{\alpha}\Gamma_{\alpha\beta\mu\nu}\psi^{\beta}A^{\mu}A^{\nu}.\end{aligned} \tag{112}$$

From the electromagnetic vertex in this Lagrangian we obtain the electromagnetic transition current in momentum space as

$$j_{\mu}(p',p) = \overline{u}^{\alpha}(p')[-\Gamma_{\alpha\beta\nu\mu}p'^{\nu} - \Gamma_{\alpha\beta\mu\nu}p^{\nu}]u^{\beta}(p), \tag{113}$$

where we wrote the free-particle spinors as $\psi^{\beta}(x) = u^{\beta}(p)e^{-ip\cdot x}$. In order to perform the analogous to the Gordon decomposition for spin-$\frac{1}{2}$ we write this current in terms of the four-momentum transfer, q, and the summed up four-momenta, k,

$$q = p' - p, \qquad k = p' + p, \tag{114}$$

to obtain

$$j_{\mu}(p',p) = \overline{u}^{\alpha}(p')[-\Gamma^{S}_{\alpha\beta\mu\nu}k^{\nu} + \Gamma^{A}_{\alpha\beta\mu\nu}q^{\nu}]u^{\beta}(p), \tag{115}$$

where $\Gamma^{S}_{\alpha\beta\mu\nu}$, and $\Gamma^{A}_{\alpha\beta\mu\nu}$ stand for the symmetric and antisymmetric parts under the $\mu \leftrightarrow \nu$ interchanging, respectively,

$$\Gamma^{S}_{\alpha\beta\mu\nu} = \frac{1}{2}(\Gamma_{\alpha\beta\mu\nu} + \Gamma_{\alpha\beta\nu\mu}), \tag{116}$$

$$\Gamma^{A}_{\alpha\beta\mu\nu} = \frac{1}{2}(\Gamma_{\alpha\beta\mu\nu} - \Gamma_{\alpha\beta\nu\mu}). \tag{117}$$

It is worth to remark that as long as the tensor $\Gamma_{\alpha\beta\mu\nu}$ is contracted with the symmetric term $p^{\mu}p^{\nu}$ in the free equation of motion (36), (37), only the symmetric part of this tensor is uniquely determined by the Poincaré projector. In contrast to this, the antisymmetric part remains ambiguous. This insight is crucial for the interacting theory since it is precisely that very anti-symmetric part that provides essential contributions to the electromagnetic couplings. As a first step in the elucidation of the electromagnetic interactions of an elementary spin-$\frac{3}{2}$ particle, we elaborate the interacting theory for the tensor in eq. (38). A straightforward calculation yields

$$\begin{aligned}\Gamma^{S}_{\alpha\beta\mu\nu} &= g_{\alpha\beta}g_{\mu\nu} - \frac{2}{3}(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha})\\ &\quad+\frac{1}{6}[(g_{\mu\alpha}\gamma_{\nu} + g_{\nu\alpha}\gamma_{\mu})\gamma_{\beta}\\ &\quad+\gamma_{\alpha}(g_{\mu\beta}\gamma_{\nu} + g_{\nu\beta}\gamma_{\mu})]\\ &\quad-\frac{1}{3}\gamma_{\alpha}\gamma_{\beta}g_{\mu\nu},\end{aligned} \tag{118}$$

$$\begin{aligned}\Gamma^{A}_{\alpha\beta\mu\nu} &= \frac{1}{3}\left[g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha} - \frac{i}{2}g_{\alpha\beta}\sigma_{\mu\nu}\right]\\ &= -\frac{i}{3}(M_{\mu\nu})_{\alpha\beta}\,,\end{aligned} \tag{119}$$

where $(M_{\mu\nu})_{\alpha\beta}$ stand for the (homogeneous) Lorentz group generators in the vector spinor representation as given in eq. (A.9). Now, in a perturbative calculation one can use the constraints for the spin-$\frac{3}{2}$ fields in the $(\frac{3}{2} - \frac{3}{2} - \gamma)$ vertex and obtain the following Gordon decomposition for the transition current:

$$\begin{aligned}j_{\mu}(p',p) &= \overline{u}^{\alpha}(p')[-g_{\alpha\beta}(p'+p)_{\mu} - \frac{i}{3}(M_{\mu\nu})_{\alpha\beta}q^{\nu}\\ &\quad+\frac{2}{3}(g_{\mu\alpha}p'_{\beta} + g_{\mu\beta}p_{\alpha})]u^{\beta}(p).\end{aligned} \tag{120}$$

According to the latter equation the gyromagnetic ratio is $g_{\frac{3}{2}} = \frac{1}{3}$. It can be shown that same value is confirmed by the gauged equation

$$[\Gamma_{\alpha\beta\mu\nu}\pi^\mu\pi^\nu - m^2\ g_{\alpha\beta}]\psi^\beta = 0, \tag{121}$$

with the tensors in eqs. (38), (116), and (117). However, the wave fronts of this gauged equation when analyzed along the lines of refs. [21,22] are found to propagate non-causally. These findings may look unsatisfactory, indeed, but as we shall see below they are not to remain the last word neither on the gyromagnetic ratio, nor on the causality issue. We shall make the point that causal propagation and gyromagnetic ratio are interconnected and that causality requires $g_{\frac{3}{2}} = 2$. The main culprit for the severe underestimation of the gyromagnetic ratio by eq. (120) is the incompleteness of $\Gamma^A_{\alpha\beta\mu\nu}$ in eq. (119) as provided by the space-time invariants. The correct value of the gyromagnetic ratio is fixed by Weinberg's theorem which states that a well-behaved forward Compton scattering amplitude for a non-strongly interacting particle with spin $s > \frac{1}{2}$ requires its gyromagnetic factor to equal $g_s = 2$ [28]. The particular case of the W-boson is instructive in that regard because this particle satisfies Weinberg's principle. Indeed, while the standard model predicts for the W-boson $g_s = 2$, the naive $U(1)_{em}$ gauging of Proca's equation yields $g_s = 1$. The difference between these two values is accounted for by additional contributions coming from the full non-Abelian $SU(2)_I \otimes U(1)_Y$ gauge structure in combination with the spontaneous breaking of the electroweak gauge symmetry. On the other hand, more recently, it was also shown that the tree-level value of the gyromagnetic ratio of the ρ^+-meson is fixed to 2 by self-consistency of the corresponding effective quantum field theory [29].

Below we shall show how to take advantage of the ambiguities of the anti-symmetric part of the $\Gamma_{\alpha\beta\mu\nu}$-tensor and construct a Lagrangian and associated wave equation such that

- the spin-$\frac{3}{2}$ particle is coupled to the electromagnetic field by a gyromagnetic factor of $g_{\frac{3}{2}} = 2$,
- the wavefronts of the solutions of the gauged equation propagate causally.

5.1 Gauged spin-$\frac{3}{2}$ equation

To begin with we first notice that the most general anti-symmetric tensor allowed by Lorentz covariance is given by

$$\begin{aligned}\widetilde{\Gamma}^A_{\alpha\beta\mu\nu} = &-i\left[g\frac{\sigma_{\mu\nu}}{2}g_{\alpha\beta} + ig'(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})\right]\\ &+ic(g_{\alpha\mu}\sigma_{\beta\nu} - g_{\alpha\nu}\sigma_{\beta\mu}) + id(\sigma_{\alpha\mu}g_{\beta\nu} - \sigma_{\alpha\nu}g_{\beta\mu})\\ &+if\ \varepsilon_{\alpha\beta\mu\nu}\gamma^5,\end{aligned} \tag{122}$$

where g, g', c, d, and f are arbitrary parameters. As a consequence, there exist infinitely many equivalent free particle theories differing by the values of the above parameters. However, upon gauging, all these equivalent free particle descriptions will become distinguishable through the different values of the multipole couplings of the spin-$\frac{3}{2}$ particle to the photon field. Only one of those coupled theories will correspond to the physical reality. The covariant projector in eq. (121) with $\Gamma_{\alpha\beta\mu\nu}$ from eq. (38) hits a $\widetilde{\Gamma}^A_{\alpha\beta\mu\nu}$ (the $\Gamma^A_{\alpha\beta\mu\nu}$ in eq. (119)) that corresponds to the particular parameter set $g = g' = \frac{1}{3}$, and $c = d = f = 0$. According to our analysis, this parameter set fails both in the description of the gyromagnetic ratio as dictated by Weinberg's theorem and in providing causal propagation. We shall remove this shortcoming in choosing an appropriate $\widetilde{\Gamma}^A_{\alpha\beta\mu\nu}$ that *ensures causal propagation of the wavefronts of the solutions of eq.* (121) *within an electromagnetic environment*. In the following, we shall also adopt $f = 0$ for simplicity. Then, we notice that hermiticity requires $c = -d$.

Back to the symmetric $\Gamma^S_{\alpha\beta\mu\nu}$-tensor, we observe that the indices α and β have to be moved to the very left, and the very right, respectively, in order to work in $\pi\cdot\psi$, and $\gamma\cdot\psi$ into the wave equation. In so doing, one finds various terms in $\Gamma^S_{\alpha\beta\mu\nu}$ that contain the electromagnetic tensor according to

$$\begin{aligned}\Gamma^S_{\alpha\beta\mu\nu}\pi^\mu\pi^\nu = &\pi^2 g_{\alpha\beta} + \frac{1}{3}\left(\gamma_\alpha\ \not{\pi} - 4\pi_\alpha\right)\pi_\beta\\ &+\frac{1}{3}(\pi_\alpha\ \not{\pi} - \gamma_\alpha\pi^2) + \frac{2}{3}ieF_{\alpha\beta}\\ &+\frac{ie}{6}\gamma_\alpha\gamma^\mu F_{\beta\mu} + \frac{ie}{6}\gamma^\mu F_{\mu\alpha}\gamma_\beta.\end{aligned} \tag{123}$$

whereas the anti-symmetric tensor yields

$$\begin{aligned}\widetilde{\Gamma}^A_{\alpha\beta\mu\nu}\ \pi^\mu\ \pi^\nu = &-i\left[g\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2}g_{\alpha\beta} - eg'F_{\alpha\beta}\right] - ie2cF_{\alpha\beta}\\ &+\ c(\pi_\alpha\ \not{\pi} - \not{\pi}\pi_\alpha)\gamma_\beta + c\gamma_\alpha(\not{\pi}\pi_\beta - \pi_\beta\ \not{\pi}).\end{aligned} \tag{124}$$

Putting all together as

$$[\widetilde{\Gamma}_{\alpha\beta\mu\nu}\pi^\mu\pi^\nu - m^2\ g_{\alpha\beta}]\psi^\beta = 0, \tag{125}$$

$$\widetilde{\Gamma}_{\alpha\beta\mu\nu} = \Gamma^S_{\alpha\beta\mu\nu} + \widetilde{\Gamma}^A_{\alpha\beta\mu\nu}\,, \tag{126}$$

results in the following new gauged equation:

$$\begin{aligned}&\Bigg((\pi^2 - m^2)g_{\alpha\beta}\\ &-i\left[g\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2}g_{\alpha\beta} - e\left(g' - 2c + \frac{2}{3}\right)F_{\alpha\beta}\right]\\ &+\frac{1}{3}\left(\gamma_\alpha\ \not{\pi} - 4\pi_\alpha\right)\pi_\beta + \frac{1}{6}(2\pi_\alpha\ \not{\pi} - 2\gamma_\alpha\pi^2)\gamma_\beta\\ &+ie\left(\frac{1}{6} - c\right)\gamma^\mu F_{\mu\alpha}\gamma_\beta + ie\left(\frac{1}{6} - c\right)\gamma_\alpha\gamma^\mu F_{\beta\mu}\Bigg)\psi^\beta\\ &= 0.\end{aligned} \tag{127}$$

The next physical consideration allows to fix the c-parameter in $\Gamma^A_{\alpha\beta\mu\nu}$ and refers to the suppression of the $\frac{3}{2} \leftrightarrow \frac{1}{2}$ transitions $\gamma\cdot\psi \leftrightarrow \psi^\mu$ (as required by perturbative calculations) through nullifying the $ie\gamma^\mu F_{\mu\alpha}\gamma_\beta$-, and

$\gamma_\alpha\gamma^\mu F_{\beta\mu}$ terms, respectively. In this manner c is fixed to $c = \frac{1}{6}$ and one is left with the two parameters g and g' which in turn describe the gyromagnetic ratios of the fermion and vector part of ψ^β. As long as one wishes to have a spin-$\frac{3}{2}$ coupled by a gyromagnetic factor of $g_{\frac{3}{2}}$ and given by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{mag}} &\equiv -\frac{eg_{\frac{3}{2}}}{2}\bar{\psi}^\alpha (M_{\mu\nu})_{\alpha\beta}\psi^\beta F^{\mu\nu} \qquad (128)\\ &= -\frac{eg_{\frac{3}{2}}}{2}\bar{\psi}^\alpha \left(i(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) + \frac{\sigma_{\mu\nu}}{2}g_{\alpha\beta}\right) F^{\mu\nu}\psi^\beta\\ &= ig_{\frac{3}{2}}\bar{\psi}^\alpha\left(\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2}g_{\alpha\beta} - e\ F_{\alpha\beta}\right)\psi^\beta, \qquad (129)\end{aligned}$$

one sees that one needs equality of the spinor and vector contributions to the gyromagnetic ratio. With this in mind, from now onward we shall assume $g = g' + \frac{1}{3} \equiv g_{\frac{3}{2}}$. The corresponding one-parameter equation for an interacting spin-$\frac{3}{2}$ particle is now given by

$$\begin{aligned}&\Bigg(\left(\pi^2 - m^2\right) g_{\alpha\beta} - ig_{\frac{3}{2}}\left(\frac{\sigma_{\mu\nu}\pi^\mu\pi^\nu}{2}g_{\alpha\beta} - e\ F_{\alpha\beta}\right)\\ &\quad +\frac{1}{3}\left(\gamma_\alpha\ \not\pi - 4\pi_\alpha\right)\pi_\beta\\ &\quad +\frac{1}{3}(\pi_\alpha\ \not\pi - \gamma_\alpha\pi^2)\gamma_\beta\Bigg)\psi^\beta = 0. \qquad (130)\end{aligned}$$

We shall fix the $g_{\frac{3}{2}}$-parameter from the requirement on causality. Before this, we notice that equations like (130) are not genuine because neither the field component ψ_0 nor its time-like momentum, π_0, ever occur. This behavior reflects the presence of constraints in eq. (130). In order to produce a genuine wave equation, one needs to obtain first the gauged constraints and back-substitute them into eq. (130). In subsequently contracting eq. (130) by γ_β, and π_β one obtains the gauged auxiliary conditions as

$$\gamma\cdot\psi = \frac{ie}{6m^2}\left(3g_{\frac{3}{2}} + 2\right)\left(F_{\mu\beta}\gamma^\mu + i\gamma^5\gamma^\alpha\tilde{F}_{\beta\alpha}\right)\psi^\beta, \quad (131)$$

and

$$\begin{aligned}m^2\pi\cdot\psi = &\Bigg(ie\left(1 - \frac{g_{\frac{3}{2}}}{2}\right)(F_{\beta\mu}\pi^\mu + \pi^\mu F_{\beta\mu}) + ieg_{\frac{3}{2}}\pi^\alpha F_{\alpha\beta}\\ &-e\left(\frac{g_{\frac{3}{2}}}{4} + \frac{1}{6}\right)\gamma^5[\gamma^\alpha\widetilde{F}_{\beta\alpha}, \not\pi] + ie\left(\frac{g_{\frac{3}{2}}}{4} - \frac{1}{6}\right)\\ &\times\{\gamma^\alpha F_{\beta\alpha}, \not\pi\}\Bigg)\psi^\beta + ie\Bigg(\left(\frac{g_{\frac{3}{2}}}{4} - \frac{1}{6}\right)\\ &\times\gamma^\nu(F_{\nu\mu}\pi^\mu + \pi^\mu F_{\nu\mu})\Bigg)\gamma\cdot\psi, \qquad (132)\end{aligned}$$

respectively. The resulting equation is now genuine and the wavefronts of its solutions would propagate causally provided the so-called characteristic determinant of the matrix that contains only the highest derivatives when replaced by n_μ, the normal vectors to the characteristic surfaces, nullifies only for real values of n_0[1]. The Velo-Zwanziger problem arises because the characteristic determinant of the (genuine) Rarita-Schwinger equation allows for n_0-roots that can become imaginary for sufficiently strong electromagnetic fields.

[1] We here follow the calculation patterns of refs. [21] and [22].

5.2 Causal propagation and gyromagnetic ratio

The expression for the matrix that provides the characteristic determinant, denoted by $D(n, g_{\frac{3}{2}})$, of eqs. (130) with the substituted eqs. (131) and (132) is now obtained as

$$\begin{aligned}D(n, g_{\frac{3}{2}}) &= |\mathcal{M}_{\alpha\beta}|,\\ \mathcal{M}_{\alpha\beta} &= n^2 g_{\alpha\beta} + \frac{1}{3}\left(\gamma_\alpha\ \not n - 4n_\alpha\right)N_\beta\\ &\quad +\frac{1}{3}\left(n_\alpha\ \not n - \gamma_\alpha n^2\right)\Gamma_\beta,\\ \Gamma_\beta &= \frac{ie}{6m^2}\left(3g_{\frac{3}{2}} + 2\right)\left(F_{\mu\beta}\gamma^\mu + i\gamma^5\gamma^\mu\tilde{F}_{\mu\alpha}\right),\\ N_\beta &= \frac{1}{m^2}\Bigg(ie\left(\frac{5}{3} - \frac{3}{2}g_{\frac{3}{2}}\right)F_{\beta\mu}n^\mu - e\left(\frac{g_{\frac{3}{2}}}{4} + \frac{1}{6}\right)\\ &\quad \times\gamma^5[\gamma^\alpha\widetilde{F}_{\beta\alpha}, \not n]\Bigg)\\ &\quad +\frac{ie}{m^2}\left(\frac{g_{\frac{3}{2}}}{2} - \frac{1}{3}\right)\gamma^\nu F_{\nu\mu}n^\mu\Gamma_\beta. \qquad (133)\end{aligned}$$

The covariant form of the characteristic determinant is now calculated to give

$$\begin{aligned}D(n, g_{\frac{3}{2}}) = (n^2)^{12}&\Bigg(\Bigg[n^2 - k^2\left(\frac{5g_{\frac{3}{2}} - 2}{4}\right)^2 (n\cdot F)^2\\ &+k^2\left(\frac{3g_{\frac{3}{2}} + 2}{4}\right)^2\left(n\cdot\tilde{F}\right)^2\Bigg]^2\\ &+\frac{k^2}{4}\left(\frac{3g_{\frac{3}{2}} + 2}{4}\right)^2\left(\frac{5g_{\frac{3}{2}} - 2}{4}\right)^2\left(F\cdot\tilde{F}\right)^2\left(n^2\right)^2\Bigg)\\ &\times\Bigg(\Bigg[n^2 + k^2\left(\frac{3g_{\frac{3}{2}} + 2}{4}\right)^2\left[\left(n\cdot\tilde{F}\right)^2 - (n\cdot F)^2\right]\Bigg]^2\\ &+\frac{k^4}{4}\left(\frac{3g_{\frac{3}{2}} + 2}{4}\right)^2\left(F\cdot\tilde{F}\right)^2\left(n^2\right)^2\Bigg). \qquad (134)\end{aligned}$$

Here, $(n\cdot F)^\nu = n_\mu F^{\mu\nu}$, $(n\cdot\tilde{F})^\nu = n_\mu\tilde{F}^{\mu\nu}$, $F\cdot\tilde{F} = F_{\mu\nu}\tilde{F}^{\mu\nu}$, and $k = \frac{2e}{3m^2}$. It is quite instructive to compare eq. (134) to the characteristic determinant of the Rarita-Schwinger equation reported in ref. [21] as

$$D(n) = \left(n^2\right)^4\left[n^2 + k^2\left(\widetilde{F}\cdot n\right)^2\right]^4. \qquad (135)$$

The advantage of eq. (134) over eq. (135) is that in the former case it is possible to factorize $(n^2)^{16}$ in the expression in the brackets on the cost of fixing $g_{\frac{3}{2}}$ to either 0 or 2, while in the latter such results impossible. To be specific, using

$$(n \cdot \widetilde{F})^2 - (n \cdot F)^2 = -\frac{1}{2} n^2 F \cdot F, \tag{136}$$

and for $g_{\frac{3}{2}} = 0, 2$ the characteristic determinant takes the following factorized form:

$$D(n, g_{\frac{3}{2}} = 0, 2) = (n^2)^{16} ((1 - 2k^2 F \cdot F)^2 + (2k^2 F \cdot \widetilde{F})^2)^2. \tag{137}$$

Thus for $g_{\frac{3}{2}} = 0, 2$, the determinant nullifies only for real and field-independent n_0-values given by

$$n_0 = \pm \sqrt{\mathbf{n}^2}, \tag{138}$$

and of multiplicity 16 each, yielding causal propagation. The vanishing gyromagnetic ratio can be associated with neutral, and $g_{\frac{3}{2}} = 2$ with charged particles. Compared to this, only eight, *i.e.* half, of the roots of the characteristic RS determinant are necessarily real and given by $n_0 = \sqrt{\mathbf{n}^2}$. In order to find the other eight roots it is first quite useful to write explicitly the four-vector $(n \cdot \widetilde{F})^\nu$ as

$$\left(n \cdot \widetilde{F}\right)^\nu = (\mathbf{B} \cdot \mathbf{n}, n_0 \mathbf{B} - \mathbf{n} \times \mathbf{E}). \tag{139}$$

Substitution in eq. (135) amounts to the following second order equation for n_0:

$$\begin{aligned} &n_0^2(1 - k^2 \mathbf{B}^2) - \mathbf{n}^2 + k^2 \mathbf{B} \cdot \mathbf{n} \\ &\quad + k^2 \left(2 n_0 \mathbf{B} \cdot (\mathbf{n} \times \mathbf{E}) - (\mathbf{n} \times \mathbf{E})^2\right) = 0. \end{aligned} \tag{140}$$

As long as the discriminant of the latter equation is frame dependent, there are frames where it can become negative and the roots imaginary. The frame dependence of eq. (140) also shows up in the possibility of finding frames where the signal velocity is superluminal, a problem first addressed by Velo and Zwanziger in refs. [21,22].

> The decisive advantage of eq. (130) over the gauged Rarita-Schwinger equation is the field and therefore frame independence of the n_0-roots, a behavior which allows for hyperbolicity of the wave equation, causal signal propagation, and a gyromagnetic ratio in accord with the requirements of unitarity in the ultra-relativistic limit.

The final form of the $\widetilde{\Gamma}_{\alpha\beta\mu\nu}$-tensor now reads

$$\begin{aligned} \widetilde{\Gamma}_{\alpha\beta\mu\nu} &= \Gamma^S_{\alpha\beta\mu\nu} - i\left(\sigma_{\mu\nu} g_{\alpha\beta} + i\frac{5}{3}(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu})\right) \\ &\quad + i\frac{1}{6}(g_{\alpha\mu}\sigma_{\beta\nu} - g_{\alpha\nu}\sigma_{\beta\mu}) - i\frac{1}{6}(\sigma_{\alpha\mu} g_{\beta\nu} - \sigma_{\alpha\nu} g_{\beta\mu}), \end{aligned} \tag{141}$$

with $\Gamma^S_{\alpha\beta\mu\nu}$ from eq. (118). It is important to emphasize that also this tensor satisfies eq. (48) meaning that the free theory remains the same as the one related to eq. (38).

Finally, a comment is in place on the hermiticity of the equation under discussion. Notice that upon substituting the gauged auxiliary conditions from eqs. (131) and (132) into eq. (130) one does not find a Hermitian equation. Above we gave the causality proof for precisely that very case for the sake of simplicity of the expressions and without any loss of generality because the proof goes through also upon making the equation Hermitian. In conclusion, the covariant spin-$\frac{3}{2}$ and mass-m projector method elaborated here hits the right way toward the consistent description of spin-$\frac{3}{2}$ within ψ_μ.

6 Summary

In this paper we developed a spin-$\frac{3}{2}$ description on the basis of the Poincaré covariant mass-m and spin-$\frac{3}{2}$ projectors in ψ_μ and explicitly worked out the corresponding Lagrangian and wave equation. Our suggested solution to the problem of the covariant and consistent description of spin-$\frac{3}{2}$ coupled to an electromagnetic field is the fully covariant second-order equation (36), (37), gauged (130) and with the tensor $\Gamma_{\alpha\beta\mu\nu}$ given in eq. (141). We studied the symmetries of the suggested Lagrangian in the massless limit and their extrapolation to the massive case, where they gave rise to constraints and introduced parameter dependence of the off-mass shell propagators. We observed that the off-shell massive spin-$\frac{3}{2}$ propagator suggested by us is of the type of propagators that appear in the massive gauge theories. From that we concluded that its parameter dependence is actually brought about by the symmetries of the massless theory, one of them being the gauge freedom, and as such can be handled by means of "gauge" fixing. We introduced electromagnetic interactions into the theory and showed that the wavefronts of the solutions of the gauged equation propagate causally provided the gyromagnetic factor of the spin-$\frac{3}{2}$ particle were to be $g_{\frac{3}{2}} = 2$ in accord with the requirements of unitarity in the ultrarelativistic limit. In this way causality calls for a $g_{\frac{3}{2}} = 2$ value of a spin-$\frac{3}{2}$ particle that is not a non-Abelian gauge field. In case the spin-$\frac{3}{2}$ particle were to be a gauge field, one would expect non-Abelian corrections to $g_{\frac{3}{2}}$ to become relevant too, much alike the case of the spin-1 electroweak bosons as reviewed by Barry Holstein in ref. [38]. Such a new type of a particle will obviously require its own appropriate $\widetilde{\Gamma}^A_{\alpha\beta\mu\nu}$-tensor with parameters that allow for accommodating the non-Abelian corrections. The gyromagnetic ratio resulting from the framework presented here, in not following Belinfante's $g_s = 1/s$ prescription, supports our claim on the different physical content of the covariant projector formalism in comparison to the traditional ones. Our approach rather points on $g_s = 2$ for any spin and is in accord with Weinberg's theorem.

The gyromagnetic ratio of $g_{\frac{3}{2}} = 2$ in combination with the causal propagation and the structure of the off-mass shell propagator, seem to qualify the formalism elaborated here as a promising candidate for the consistent description of massive spin-$\frac{3}{2}$ fields.

Work supported by Consejo Nacional de Ciencia y Tecnologia (CONACyT) Mexico under projects 37234-E and C01-39820.

Appendix A.

In this appendix we collect conventions and some results on the symmetry of space-time under rotations, boost and translations transformations that constitute the Poincaré group for which the squared Pauli-Lubanski vector is a Casimir invariant. In terms of the Poincaré group generators, $M_{\mu\nu}$ and p_η and their algebra [39]

$$\begin{aligned}[M_{\mu\nu}, M_{\alpha\beta}] &= -i(g_{\mu\alpha}M_{\nu\beta} - g_{\mu\beta}M_{\nu\alpha} + g_{\nu\beta}M_{\mu\alpha} \\ &\quad -g_{\nu\alpha}M_{\mu\beta}), \\ [M_{\alpha\beta}, p_\mu] &= -i(g_{\mu\alpha}p_\beta - g_{\mu\beta}p_\alpha), \\ [p_\mu, p_\nu] &= 0, \end{aligned} \tag{A.1}$$

where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor, the Pauli-Lubanski (PL) vector is defined as

$$\mathcal{W}_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M^{\nu\alpha}p^\beta, \tag{A.2}$$

with $\epsilon_{0123} = 1$. This operator can be shown to satisfy the following commutation relations:

$$\begin{aligned}[M_{\mu\nu}, \mathcal{W}_\alpha] &= -i(g_{\alpha\mu}\mathcal{W}_\nu - g_{\alpha\nu}\mathcal{W}_\mu), \quad [\mathcal{W}_\alpha, p_\mu] = 0, \\ [\mathcal{W}_\alpha, \mathcal{W}_\beta] &= -i\epsilon_{\alpha\beta\mu\nu}\mathcal{W}^\mu p^\nu, \end{aligned} \tag{A.3}$$

i.e. it transforms as a four-vector under Lorentz transformations. Moreover, its square commutes with all the generators and is a group invariant. For this reason, elementary particles are required to transform invariantly under the action of $\mathcal{W}^2$ and to be labeled by the $\mathcal{W}^2$ eigenvalues next to those of p^2.

In general, for a specific representation, the generators of the (homogeneous) Lorentz group $M_{\mu\nu}$ carry additional indices. We denote these indices by capital Latin letters as $(M_{\mu\nu})_{AB}$. The Lorentz group generators in the vector (*i.e.* $(\frac{1}{2}, \frac{1}{2})$) and spinor (*i.e.* $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$) space are respectively given as

$$(M_V^{\rho\sigma})_{\alpha\beta} = i(g^\rho{}_\alpha g^\sigma{}_\beta - g^\rho{}_\beta g^\sigma{}_\alpha), \qquad (M_S^{\rho\sigma})_{ab} = \frac{1}{2}(\sigma^{\rho\sigma})_{ab}. \tag{A.4}$$

The indices A, B are Lorentz indices (in the vector basis) for the vector space: $A = \{\alpha\}, B = \{\beta\}$, whereas for the spinor representation they are spinorial indices: $A = \{a\}, B = \{b\}$. The Pauli-Lubanski operators in vector and spinor space, denoted respectively by W^λ, w^λ, have the explicit form

$$[W_\lambda]_{\alpha\beta} = i\epsilon_{\lambda\alpha\beta\mu}p^\mu, \qquad (w_\lambda)_{ab} = \frac{i}{2}(\gamma_5\sigma_{\lambda\nu})_{ab}p^\nu. \tag{A.5}$$

The squared Pauli-Lubanski operators are now calculated as

$$[W^2]_{\alpha\beta} = -2\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)p^\mu p^\nu, \tag{A.6}$$

$$[w^2]_{ab} = -\frac{1}{4}(\sigma_{\lambda\mu})_{ac}(\sigma^\lambda{}_\nu)_{cb}p^\mu p^\nu. \tag{A.7}$$

In particular, our general equation (20) for the spin-1 subspace in $(\frac{1}{2}, \frac{1}{2})$ reads

$$[W^2]_\alpha{}^\beta A_\beta = -2m^2 A_\alpha. \tag{A.8}$$

As for the vector spinor space, the (homogeneous) Lorentz group generators are given by

$$(M^{\rho\sigma})_{\alpha\beta ab} = (M_V^{\rho\sigma})_{\alpha\beta}\,\delta_{ab} + g_{\alpha\beta}(M_S^{\rho\sigma})_{ab}. \tag{A.9}$$

The Pauli-Lubanski vector for the vector spinor representation reads

$$(\mathcal{W}^\lambda)_{\alpha a\beta b} = (W^\lambda)_{\alpha\beta}\delta_{ab} + g_{\alpha\beta}(w^\lambda)_{ab}. \tag{A.10}$$

The indices A, B in this case correspond to the sets $A = \{\alpha a\}, B = \{\beta b\}$. The squared Pauli-Lubanski operator in the vector spinor representation reads

$$\begin{aligned}(\mathcal{W}^2)_{\alpha\beta ab} &= (W^2)_{\alpha\beta}\delta_{ab} + (W)_{\alpha\beta}\cdot(w)_{ab} + (w)_{ab}\cdot(W)_{\alpha\beta} \\ &\quad + g_{\alpha\beta}(w^2)_{ab}. \end{aligned} \tag{A.11}$$

We obtain the involved operators as

$$\begin{aligned}(W^2)_{\alpha\beta} &= -2\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)p^\mu p^\nu, \\ (w^2)_{ab} &= -\frac{1}{4}(\sigma_{\lambda\mu}\sigma^\lambda{}_\nu)_{ab}\;p^\mu p^\nu, \\ (W\cdot w + w\cdot W)_{\alpha a\beta b} &= -\frac{1}{2}(\epsilon^\lambda{}_{\alpha\beta\mu}\gamma^5(\sigma_{\lambda\nu})_{ab} \\ &\quad + \epsilon^\lambda{}_{\alpha\beta\nu}\gamma^5(\sigma_{\lambda\mu})_{ab})\;p^\mu p^\nu. \end{aligned} \tag{A.12}$$

In substituting eqs. (A.12) into (A.11) results in

$$(\mathcal{W}_\lambda\mathcal{W}^\lambda)_{\alpha\beta ab} = T_{\alpha a\beta b\mu\nu}p^\mu p^\nu, \tag{A.13}$$

with

$$\begin{aligned}T_{\alpha a\beta b\mu\nu} &= -2\left(g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right)\delta_{ab} - \frac{1}{4}g_{\alpha\beta}(\sigma_{\lambda\mu}\sigma^\lambda{}_\nu)_{ab} \\ &\quad -\frac{1}{2}(\epsilon^\lambda{}_{\alpha\beta\mu}\gamma^5(\sigma_{\lambda\nu})_{ab} + \epsilon^\lambda{}_{\alpha\beta\nu}\gamma^5(\sigma_{\lambda\mu})_{ab}). \end{aligned} \tag{A.14}$$

Appendix B.

In this appendix we construct the solutions of the free particle equation (40) for the massive case in the basis of the direct products of the polarization four-vectors $\epsilon^\mu(\mathbf{p}, s)$ and generic $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ spinors, denoted by $w(\mathbf{p}, \sigma)$, where $s = \pm 1, 0$ ($\sigma = \pm\frac{1}{2}$) stands for the spin-1 (spin-$\frac{1}{2}$) magnetic quantum number. The solutions to the wave equation are written as

$$\psi^\beta(x) = w^\beta(\mathbf{p}, \lambda)e^{\pm ip\cdot x}. \tag{B.1}$$

The vector spinors $w^\beta(\mathbf{p}, \lambda)$ satisfy the equation

$$K_{\alpha\beta}(p)w^\beta(\mathbf{p}, \lambda) = m^2 w_\alpha(\mathbf{p}, \lambda), \tag{B.2}$$

where λ denotes the spin-$\frac{3}{2}$ magnetic quantum number. The solutions are constructed from the coupling between the vector and the spinor representations in momentum space using the coefficients of Clebsh-Gordan of the little group to the Lorentz group which for massive particles is $SU(2)$, the universal covering of O(3),

$$w^{\beta}(\mathbf{p},\lambda)=\sum_{s,\sigma}\left\langle 1,s;\frac{1}{2},\sigma\middle|1,\frac{1}{2};\frac{3}{2},\lambda\right\rangle\ \epsilon^{\beta}(\mathbf{p},s)\ w(\mathbf{p},\sigma). \tag{B.3}$$

The specific combinations are:

$$\begin{aligned}
w^{\beta}\left(\mathbf{p},\frac{3}{2}\right)&=\epsilon^{\beta}(\mathbf{p},1)\ w\left(\mathbf{p},\frac{1}{2}\right), \qquad\qquad (B.4)\\
w^{\beta}\left(\mathbf{p},\frac{1}{2}\right)&=\sqrt{\frac{1}{3}}\epsilon^{\beta}(\mathbf{p},1)\ w\left(\mathbf{p},-\frac{1}{2}\right)\\
&\quad+\sqrt{\frac{2}{3}}\epsilon^{\beta}(\mathbf{p},0)\ w\left(\mathbf{p},\frac{1}{2}\right),\\
w^{\beta}\left(\mathbf{p},-\frac{1}{2}\right)&=\sqrt{\frac{1}{3}}\epsilon^{\beta}(\mathbf{p},-1)\ w\left(\mathbf{p},\frac{1}{2}\right)\\
&\quad+\sqrt{\frac{2}{3}}\epsilon^{\beta}(\mathbf{p},0)\ w\left(\mathbf{p},-\frac{1}{2}\right),\\
w^{\beta}\left(\mathbf{p},\frac{3}{2}\right)&=\epsilon^{\beta}(\mathbf{p},-1)\ w\left(\mathbf{p},-\frac{1}{2}\right).
\end{aligned}$$

It is important to be aware of the fact that the basis vectors in any irreducible representation of the Lorentz group are defined exclusively by the Lorentz group generators and can be constructed straightforwardly from the respective algebra and without any reference to a wave equation. Only afterward can one entertain (or, construct from the group invariants as done in this work) various wave equations that have as solutions the states of interest and which are fully or partially consistent with the Lorentz group structure. With this in mind, any wave equation in momentum space that claims to describe spin-$\frac{3}{2}$ in ψ_{μ}, be it the Rarita-Schwinger framework, be it the approach presented here, or, any other equation, necessarily hits the generic four-vector spinors in eqs. (B.4), modulo, as we shall see below, possible differences in the basis choice in the spinor space. Our main point here is that compared to the Rarita-Schwinger formalism, our equation (40), being built up systematically from the Casimir invariants of the Lorentz and the Poincaré groups, provides a better tool for the description of particles coupled to external electromagnetic fields. Nonetheless, it is instructive to highlight the construction of the solutions to eq. (40) in order to discuss the number of degrees of freedom upon passing to coordinate space.

The specific example under consideration, the four-vector space, has been studied along the above line, among others, in ref. [40]. There, one finds the explicit expressions for the polarization vectors (up to notational differences) as

$$\epsilon^{\beta}(\mathbf{p},+1)=-\frac{1}{\sqrt{2}m(p^0+m)}\begin{pmatrix}(p^0+m)p^+\\ m(p^0+m)+p^1p^+\\ i[m(p^0+m)-ip^2p^+]\\ p^3p^+\end{pmatrix}, \tag{B.5}$$

$$\epsilon^{\beta}(\mathbf{p},0)=\frac{1}{m(p^0+m)}\begin{pmatrix}(p^0+m)p^3\\ p^3p^1\\ p^3p^2\\ m(p^0+m)+(p^3)^2\end{pmatrix}, \tag{B.6}$$

and

$$\epsilon^{\beta}(\mathbf{p},-1)=\frac{1}{\sqrt{2}m(p^0+m)}\begin{pmatrix}(p^0+m)p^-\\ m(p^0+m)+p^1p^-\\ -i[m(p^0+m)+ip^2p^-]\\ p^3p^-\end{pmatrix}, \tag{B.7}$$

where $p^{\pm}=p^1\pm ip^2$. These states are normalized as

$$[\epsilon_{\beta}(\mathbf{p},s)]^{\dagger}\,\epsilon^{\beta}(\mathbf{p},s')=-\delta_{ss'}, \tag{B.8}$$

and reduce to the well-known $\mathbf{S}^2$–, and S_z eigenstates in the rest frame,

$$\begin{aligned}
\epsilon^{\beta}(\mathbf{0},1)=\frac{1}{\sqrt{2}}\,(0,-1,-i,0)\,,\qquad \epsilon^{\beta}(\mathbf{0},0)&=(0,0,0,1),\\
\epsilon^{\beta}(\mathbf{0},-1)&=\frac{1}{\sqrt{2}}(0,1,-i,0).
\end{aligned} \tag{B.9}$$

In order for the vector spinors in eq. (B.4) to describe spin-$\frac{3}{2}$ they have to satisfy the constraints,

$$p_{\beta}w^{\beta}(\mathbf{p},\lambda)=0,\qquad \gamma_{\beta}w^{\beta}(\mathbf{p},\lambda)=0. \tag{B.10}$$

It is easy to check that the first condition is fulfilled for all $w^{\beta}(\mathbf{p},\lambda)$ in eq. (B.4) because of $p_{\beta}\epsilon^{\beta}(\mathbf{p},s)=0$, while the second condition imposes restrictions on the spinor components. For the sake of transparency, we first solve these constraints in the rest frame. The corresponding spinors in an arbitrary frame can then be easily obtained just in boosting them by the boost operator for the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation space which can be found, *e.g.*, in [41] and is given by

$$B_S(\mathbf{p})=\frac{1}{\sqrt{2m(p^0+m)}}\begin{pmatrix}p^0+m & \boldsymbol{\sigma}\cdot\mathbf{p}\\ \boldsymbol{\sigma}\cdot\mathbf{p} & p^0+m\end{pmatrix}. \tag{B.11}$$

The condition $\gamma_{\beta}w^{\beta}(\mathbf{0},\lambda)=0$ requires the spinors to satisfy

$$\begin{aligned}
\gamma^{+}w\left(\mathbf{0},\frac{1}{2}\right)&=0,\\
\gamma^{+}\ w\left(\mathbf{0},-\frac{1}{2}\right)-\gamma^{3}\ w\left(\mathbf{0},\frac{1}{2}\right)&=0,\\
\gamma^{-}w\left(\mathbf{0},\frac{1}{2}\right)+\gamma^{3}\ w\left(\mathbf{0},-\frac{1}{2}\right)&=0,\\
\gamma^{-}w\left(\mathbf{0},-\frac{1}{2}\right)&=0,
\end{aligned} \tag{B.12}$$

where $\gamma^{\pm}\equiv\gamma^1\pm i\gamma^2$. A straightforward calculation shows that the most general form of the spinors satisfying the above constraints are

$$w\left(\mathbf{0},\frac{1}{2}\right)=\begin{pmatrix}a\\0\\b\\0\end{pmatrix},\qquad w\left(\mathbf{0},-\frac{1}{2}\right)=\begin{pmatrix}0\\a\\0\\b\end{pmatrix}, \tag{B.13}$$

where a, b are arbitrary complex parameters. Normalization requires $|a|^2 - |b|^2 = 1$. Notice that the $w(\mathbf{0}, \frac{1}{2})$- and $w(\mathbf{0}, -\frac{1}{2})$-spinor entering the construction of the spin-$\frac{3}{2}$ in eq. (B.4) are related to each other. Once we choose a specific $w(\mathbf{0}, \frac{1}{2})$-spinor, its $w(\mathbf{0}, -\frac{1}{2})$-companion is fixed. The constraints allow only two independent spinors $w(\mathbf{0}, \frac{1}{2})$ to enter the spin-$\frac{3}{2}$ construct and which in turn yield two spin-$\frac{3}{2}$ multiplets. The specific form of the latter depends on the basis chosen for $w(\mathbf{0}, \frac{1}{2})$. A natural choice is the Dirac basis and we shall use it in the following for the sake of definitiveness *but from eq.* (B.13) *it is clear that our approach does not restrict to this choice.* These two multiplets are generated by

$$w^{(1)}\left(\mathbf{0}, \frac{1}{2}\right) \equiv u\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$w^{(1)}\left(\mathbf{0}, -\frac{1}{2}\right) \equiv u\left(\mathbf{0}, -\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{(B.14)}$$

and

$$w^{(2)}\left(\mathbf{0}, \frac{1}{2}\right) \equiv v\left(\mathbf{0}, \frac{1}{2}\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$w^{(2)}\left(\mathbf{0}, -\frac{1}{2}\right) \equiv v\left(\mathbf{0}, -\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{(B.15)}$$

the $w^{(r)}(\mathbf{0}, -\frac{1}{2})$'s, with $r = 1, 2$, being dictated by the constraints according to eq. (B.13). The corresponding boosted spinors can be easily constructed as conventional Dirac spinors just in applying the boost operator in eq. (B.11). In so doing, one obtains

$$w^{(1)}\left(\mathbf{p}, \frac{1}{2}\right) = N \begin{pmatrix} p^0 + m \\ 0 \\ p^3 \\ p^+ \end{pmatrix},$$

$$w^{(1)}\left(\mathbf{p}, -\frac{1}{2}\right) = N \begin{pmatrix} 0 \\ p^0 + m \\ p^- \\ -p^3 \end{pmatrix}, \quad \text{(B.16)}$$

and

$$w^{(2)}\left(\mathbf{p}, \frac{1}{2}\right) = N \begin{pmatrix} p^3 \\ p^+ \\ p^0 + m \\ 0 \end{pmatrix},$$

$$w^{(2)}\left(\mathbf{p}, -\frac{1}{2}\right) = N \begin{pmatrix} p^- \\ -p^3 \\ 0 \\ p^0 + m \end{pmatrix}, \quad \text{(B.17)}$$

where $N = 1/\sqrt{2m(p^0 + m)}$. The $w^{(1)}(\mathbf{p}, \sigma)$-spinors are normalized to 1, and the $w^{(2)}(\mathbf{p}, \sigma)$'s to (-1), being mutually orthogonal, meaning that the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$-representation is spanned by four basis vectors, as it should be. The spin-$\frac{3}{2}$ vector spinors in eqs. (B.4) which contain the specific $w^{(1)}(\mathbf{p}, \sigma)$- and $w^{(2)}(\mathbf{p}, \sigma)$-spinors in eqs. (B.16), (B.17), will be from now onward denoted by $U^\mu(\mathbf{p}, \lambda)$, and $V^\mu(\mathbf{p}, \lambda)$, respectively. The U^μ's and V^μ's are normalized in their turn to (-1), and 1, and span a basis of the spin-$\frac{3}{2}$ Poincaré subspace in the $(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$-representation space.

In principle, we have the following 16 vector spinors:

$$\psi_U^\beta(\mathbf{p}, \lambda) = e^{\pm ip\cdot x} U^\beta(\mathbf{p}, \lambda), \quad \text{(B.18)}$$

$$\psi_V^\beta(\mathbf{p}, \lambda) = e^{\pm ip\cdot x} V^\beta(\mathbf{p}, \lambda), \quad \text{(B.19)}$$

as the solutions to

$$K_{\alpha\beta} \Psi^\beta = m^2 \Psi_\alpha. \quad \text{(B.20)}$$

However, it has to be emphasized that *no doubling occurs in momentum space.* In momentum space one encounters the correct number of degrees of freedom as required for the description of a spin-$\frac{3}{2}$ particle and anti-particle, namely eight. The doubling of the solutions occurs solely in x-space and at cost of the $\exp(\pm ip\cdot x)$ phase factor. This phase factor can be fixed upon quantization in such a way that the positive energy accompanies, say, the U^μ, while the negative one, the V^μ vector spinors. In this fashion, the quantized theory can be furnished to contain again only the required eight degrees of freedom. In order to show this, we must first construct the charge conjugation operator. It can be easily shown that if $\psi^\beta(x)$ is a solution of the equation of motion coupled to an external electromagnetic field,

$$\left[\Gamma_{\alpha\beta\mu\nu}(\partial^\mu - ieA^\mu)(\partial^\nu - ieA^\nu) + m^2 g_{\mu\nu}\right] \psi^\beta(x) = 0, \quad \text{(B.21)}$$

then the charge conjugated field,

$$\psi_c^\beta \equiv \eta_c \gamma^2 \gamma^0 \left(\overline{\psi}^\beta\right)^T, \quad \text{(B.22)}$$

with η_c a phase, satisfies eq. (B.21) but with the opposite charge, *i.e.*

$$\left[\Gamma_{\alpha\beta\mu\nu}(\partial^\mu + ieA^\mu)(\partial^\nu + ieA^\nu) + m^2 g_{\mu\nu}\right] \psi_c^\beta(x) = 0. \quad \text{(B.23)}$$

Under charge conjugation the solutions in eq. (B.1) transform as

$$\psi_c^\mu(x) = w_c^\mu(\mathbf{p}, \lambda)\, e^{\mp ip\cdot x}, \quad \text{(B.24)}$$

where

$$w_c^\mu(\mathbf{p}, \lambda) \equiv \eta_c \gamma^2 \gamma^0 \left[\bar{w}^\mu(\mathbf{p}, \lambda)\right]^T. \quad \text{(B.25)}$$

A straightforward calculation using the general spinors in eq. (B.13) yields

$$\begin{aligned} w_c^\beta\left(\mathbf{p},\frac{3}{2}\right) &= i\eta_c\left(\epsilon^\beta(\mathbf{p},-1)\ \chi\left(\mathbf{p},-\frac{1}{2}\right)\right) \\ &\equiv \chi^\beta\left(\mathbf{p},-\frac{3}{2}\right), \\ w_c^\beta\left(\mathbf{p},\frac{1}{2}\right) &= -i\eta_c\left(\sqrt{\frac{1}{3}}\epsilon^\beta(\mathbf{p},-1)\ \chi\left(\mathbf{p},\frac{1}{2}\right)\right. \\ &\quad \left.+\sqrt{\frac{2}{3}}\epsilon^\beta(\mathbf{p},0)\chi\left(\mathbf{p},-\frac{1}{2}\right)\right)\equiv\chi^\beta\left(\mathbf{p},-\frac{1}{2}\right), \\ w_c^\beta\left(\mathbf{p},-\frac{1}{2}\right) &= i\eta_c\left(\sqrt{\frac{1}{3}}\epsilon^\beta(\mathbf{p},1)\ \chi\left(\mathbf{p},-\frac{1}{2}\right)\right. \\ &\quad \left.+\sqrt{\frac{2}{3}}\epsilon^\beta(\mathbf{p},0)\ \chi\left(\mathbf{p},\frac{1}{2}\right)\right)\equiv\chi^\beta\left(\mathbf{p},\frac{1}{2}\right), \\ w_c^\beta\left(\mathbf{p},\frac{3}{2}\right) &= -i\eta_c\left(\epsilon^\beta(\mathbf{p},1)\ \chi\left(\mathbf{p},\frac{1}{2}\right)\right)\equiv\chi^\beta\left(\mathbf{p},\frac{3}{2}\right). \end{aligned}$$

where, $\chi(\mathbf{p},\sigma)$ are obtained by boosting the following general rest frame spinors:

$$\chi\left(\mathbf{0},\frac{1}{2}\right)\equiv\begin{pmatrix} -b^* \\ 0 \\ a^* \\ 0\end{pmatrix}, \qquad \chi\left(\mathbf{0},-\frac{1}{2}\right)\equiv\begin{pmatrix} 0 \\ -b^* \\ 0 \\ a^*\end{pmatrix}. \tag{B.26}$$

Notice that also the states in eq. (B.26) carry spin-$\frac{3}{2}$ and satisfy the constraints, $\gamma_\beta w_c^\beta(\mathbf{0},\lambda)=p_\beta w_c^\beta(\mathbf{0},\lambda)=0$. As such they have necessarily the same structure as those in eq. (B.4) modulo the different phase convention. That the $w^{(2)}(\mathbf{0},\pm\frac{1}{2})$ constructs act as anti-particles to $w^{(1)}(\mathbf{0},\pm\frac{1}{2})$ can be seen explicitly from eqs. (B.14), (B.15) in which case we obtain

$$U_c^\beta(\mathbf{p},\lambda)=\eta(\lambda)V^\beta(\mathbf{p},-\lambda). \tag{B.27}$$

Here, $\eta(\lambda)=i\eta_c$ for $\lambda=\frac{3}{2},-\frac{1}{2}$ and $\eta(\lambda)=-i\eta_c$ for $\lambda=-\frac{3}{2},\frac{1}{2}$. In summary, the spin-$\frac{3}{2}$ subspace of the four-vector spinor representation space has the desired eight degrees of freedom in momentum space, $\{w^{(1)\beta}(\mathbf{p},\lambda),w_c^{(1)\beta}(\mathbf{p},\lambda)\}$. If the latter states are identified as particles with the corresponding solutions being

$$\psi^\beta(\mathbf{p},\lambda)=e^{-ip\cdot x}w^{(1)\beta}(\mathbf{p},\lambda), \tag{B.28}$$

then the charge conjugated solutions are

$$\psi_c^\beta(\mathbf{p},\lambda)=e^{+ip\cdot x}w_c^{(1)\beta}(\mathbf{p},\lambda)=\eta(\lambda)e^{+ip\cdot x}w^{(2)\beta}(\mathbf{p},-\lambda). \tag{B.29}$$

Certainly, states with the phases $e^{-ip\cdot x}$ and $e^{+ip\cdot x}$ interchanged are also solutions, but this amounts just to interchanging roles between particles and antiparticles. Finally, we like to stress once more that although the standard Rarita-Schwinger spinors also satisfy eq. (40), the formalism applies to any generic spinors in $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$, and is not restricted to the Dirac spinors employed above.

As to vector spinors that describe massless particles, it has to be said that these cannot be obtained from the massive ones in taking the limit of interest. This is due to the circumstance that the little group of massless particles is the universal covering of the group of translations on the plane and not $SU(2)$ as in the case of massive particles. The Casimirs of the symmetry group of the massless theory are not limiting cases of the Casimirs of the symmetry group underlying the massive theory. If one wishes to describe massless spin-$\frac{3}{2}$ particles, such as the massless gravitino, or, possibly massless excited leptons, one needs to design the theory from scratch in starting with projectors onto maximal helicities of massless particles. This problem is not new but also inherent to the Rarita-Schwinger, or, more general, to the Fierz-Pauli formalism, where several schemes are under consideration specifically in the gravity literature [42]. The description of massless spin-$\frac{3}{2}$ particles in the spirit of the present study, *i.e.* in terms of Lagrangians that have been systematically designed from the Casimir invariants of the relevant symmetry group of the problem, is certainly a problem worth to be looked up in future research.

References

1. Particle Data Group (W.-M. Yao *et al.*), J. Phys. G **33**, 1 (2006).
2. V.D. Burkert, T.S.H. Lee, Int. J. Mod. Phys. E **13**, 1035 (2004).
3. L. Tiator, D. Drechsel, S. Kamalov, M.M. Giannini, E. Santopinto, A. Vassallo, Eur. Phys. J. A **19**, 55 (2004).
4. CSSM Lattice Collaboration (J.M. Zanotti *et al.*), Phys. Rev. D **68**, 054506 (2003).
5. Mitsuo J. Hayashi, Mod. Phys. Lett. A **16**, 2103 (2001).
6. S.I. Kruglov, Radiat. Phys. Chem. **75**, 723 (2006).
7. M. Kaku, *Quantum Field Theory* (Oxford University Press, New York-Oxford, 1993).
8. A.L. Maroto, A. Mazumdar, Phys. Rev. Lett. **84**, 1655 (2000); R. Allahverdi, A. Jokinen, A. Mazumdar, Phys. Rev. D **71**, 043505 (2005); M. Kawasaki, K. Kohri, T. Moroi, E-Print Archive: hep-ph/0410287; D.H. Lyth, Phys. Lett. B **488**, 417 (2000); H.P. Nilles, M. Peloso, L. Sorbo, Phys. Rev. Lett. **87**, 051302 (2001); R. Kallosh, L. Kofman, A. Linde, A.V. Proeyen, Phys. Rev. D **61**, 103503 (2000).
9. M. Fierz, W. Pauli, Proc. R. Soc. London, Ser. A **173**, 211 (1939).
10. W. Rarita, J. Schwinger, Phys. Rev. **60**, 61 (1941).
11. S. Weinberg, Phys. Rev. B **133**, 1318 (1964).
12. P.A. Moldauer, K.M. Case, Phys. Rev. **102**, 279 (1956).
13. V. Bernard, T.R. Hemmert, Ulf-G. Meissner, Phys. Lett. B **565**, 137 (2003); C. Hacker, N. Wies, J. Gegelia, S. Scherer, Phys. Rev. C **72**, 055203 (2005); V. Bernard, T.R. Hemmert, U.G. Meissner, Phys. Lett. B **622**, 141 (2005).
14. T.R. Hemmert, B.R. Holstein, J. Kambor, Phys. Lett. B **395**, 89 (1997); T.R. Hemmert, B.R. Holstein, J. Kambor, J. Phys. G **24**, 1831 (1998); T.R. Hemmert, talk given at the *Workshop on the Physics of Excited Nucleons (NSTAR 2001), Mainz, Germany, 7-10 Mar 2001*; E-Print Archive: nucl-th/0105051.

15. L.M. Nath, B. Etemadi, J.D. Kimel, Phys. Rev. D **3**, 2153 (1971); R. Davidson, N.C. Mukhopadhyay, R.S. Wittman, Phys. Rev. D **43**, 71 (1991).
16. M. Napsuciale, J.L. Lucio, Phys. Lett. B **384**, 227 (1996); Nucl. Phys. B **494**, 260 (1997).
17. S. Kamefuchi, L. O'Raifeartaigh, A. Salam, Nucl. Phys. **28**, 529 (1961).
18. M. El Amiri, G. Lopez Castro, J. Pestieau, Nucl. Phys. A **543**, 673 (1992).
19. M. Benmerrouche, R.M. Davidson, N.C. Mukhopadhyay, Phys. Rev. C **39**, 2339 (1989).
20. K. Johnson, E.C. Sudarshan, Ann. Phys. (N.Y.) **13**, 126 (1961).
21. G. Velo, D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
22. G. Velo, D. Zwanziger, Phys. Rev. **188**, 2218 (1969).
23. V. Pascalutsa, R. Timmermans, Phys. Rev. C **60**, 042201 (1999); V. Pascalutsa, Phys. Rev. D **58**, 096002 (1998).
24. A.F. Rañada, G. Sierra, Phys. Rev. D **22**, 2416 (1980).
25. S.I. Kruglov, Int. J. Mod. Phys. A **21**, 1143 (2006).
26. T. Fukuyama, K. Yamamoto, Prog. Theor. Phys. **49**, 304 (1973).
27. Teymour Darkhosh, Phys. Rev. D **32**, 3251 (1985).
28. S. Weinberg, in *Lectures on elementary particles and quantum field theories*, *Proceedings of the Brandeis Summer Institute, 1970*, edited by S. Deser, Vol. **1** (Brandeis University, MIT Press, Cambridge, MA, 1970).
29. D. Djukanovic, M.R. Schindler, J. Gegelia, S. Scherer, Phys. Rev. Lett. **95**, 012001 (2005).
30. J. Weda, *Spin-3/2 particles and consistent $\pi N\Delta$ and $\gamma N\Delta$-couplings*, PhD Thesis, KVI, University of Croningen, July, 1999.
31. S. Nozawa, D. Leineweber, Phys. Rev. D **42**, 3567 (1990).
32. V. Pascalutsa, Nucl. Phys. A **680**, 76 (2000).
33. J.P. Elliot, P.G. Dawber, *Symmetry in Physics*, Vol. **2** (The Macmillan Press Ltd., London, 1979) Chapt. 15.
34. M. Kirchbach, Found. Phys. **33**, 761 (2003).
35. P. Van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981).
36. V. Pascalutsa, M. Vanderhaeghen, Phys. Lett. B **636**, 31 (2006); V. Pascalutsa, R. Timmermans, Phys. Rev. C **60**, 042201 (1999).
37. I.J.R. Aitchison, A.J.G. Hey, *Gauge Theories In Particle Physics, A Practical Introduction* (Graduate Student Series in Physics, Bristol, UK, Hilger, 1982).
38. Barry R. Holstein, *How Large is the "Natural" Magnetic Moment?* E-Print Archive: hep-ph/0607187.
39. Wu-Ki-Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).
40. D.V. Ahluwalia, M. Kirchbach, Mod. Phys. Lett. A **16**, 1377 (2001); M. Kirchbach, D.V. Ahluwalia, Phys. Lett. B **529**, 124 (2002).
41. L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, 1985) p. 443.
42. G.R. Allcock, J. Phys. A: Math. Gen. **10**, 267 (1977); J. Fang, C. Frondsdal, Phys. Rev. D **18**, 3630 (1978); Steven Corley, Phys. Rev. D **59**, 086003 (1999).